THEORY OF ANISOTROPIC PLATES

S. A. AMBARTSUMYAN

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STRENGTH
STABILITY
VIBRATION

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THEORY OF ANISOTROPIC PLATES

(STRENGTH, STABILITY, AND VIBRATION)

S. A. AMBARTSUMYAN

This book is devoted to the development and formulation of an improved theory of anisotropic plates free from the basic hypothesis of the classical theory, i.e., from the hypothesis of nondeformable normals. The versions of the improved theory considered in this book are, in fact, new approaches for the consideration of the effect of transverse displacements and of normal stresses with respect to the middle-plate surface. These versions introduce a basic correction to the classical theory and may be applied in the derivation of the first correction to the basic stress state of the classical theory. The magnitude of this correction increases together with the ratio E_i/G_{i3} and may be significant for highly anisotropic plates.

The book consists of four chapters.

The first chapter presents information from the theory of elasticity of the anisotropic body which is needed for further presentation of the material. The second chapter presents two improved theories of anisotropic plates based upon the principal statements of the theory of elasticity of the anisotropic body. These theories are based on hypotheses which impose limitations only on transverse stress-deformation characteristics of the plate. The third chapter considers numerous problems regarding the determination of stresses and displacements of different types of anisotropic plates with different forms of loading. The fourth chapter is devoted to problems of stability and vibration. Free and forced vibrations, static and dynamic stability, and supersonic flutter of different types of anisotropic and transversely isotropic plates are considered here. Numerous examples show that the classical theory of anisotropic plates is imperfect in many cases and the results obtained using this theory are not always acceptable in many important applied problems. There are 24 tables, 30 figures and 98 references in this book.

PREFACE

The problem of strength, stability and vibration of plates has attracted the attention of researchers for many years.

A wide application of anisotropic materials in modern technology enhanced a particular interest among researchers regarding the theory of anisotropic plates. Numerous results obtained in this field were generalized and systematized in an important and useful monograph by S. G. Lekhnitskii titled "Anisotropic Plates", which is devoted to the classical theory of anisotropic plates.

The studies conducted during recent years on the theory of shells and plates, both in the USSR and in foreign countries, are devoted in many instances to the improvement of theories free from the basic hypothesis of the classical theory, i.e., from the hypothesis of nondeformable normals. The great interest of researchers toward new and improved theories of shells and plates is caused by the fact that the classical theory is imperfect and the results obtained in many cases are not acceptable for many important applied problems. Such problems include, for example, the problem of high frequency vibration, the propagation of elastic waves, the concentration of stresses, the consideration of plates of moderate thickness and, of course, the problem of anisotropic plates.

At the present time, despite a number of articles appearing in various journals, there is not a single book which is devoted entirely to the improved theories of anisotropic plates. The author of this book hopes that this work will fill the gap to some extent.

The book is based on studies conducted during recent years by the author himself and his co-workers. The book is devoted to the development and formulation of a theory of anisotropic plates which will define more precisely the principal stress state of the plate. But similar to the classical theory and to other improved theories of this kind, it does not produce, as a rule, a true picture with regard to the stress state of the plate near the edge. The author presents two versions of the improved theory which introduce a basic correction to the classical theory and may be applied in the derivation of the first correction in the principal stress state presented by the classical theory. The extent of this correction increases together with the ratio E_i/G_{i3} and can become significant for highly anisotropic plates.

The improved theories of anisotropic shells and plates considered in this book represent, in fact, new methods for accounting for the influences of transverse shears and normal stresses σ_z. When the problems of bending, stability and vibration of anisotropic plates are considered, these theories are not the only ones possible, as there are many approaches which can improve the theories of

shells and plates. Many intensive studies are being conducted as well as many discussions regarding the derivation of improved theories of plates and shells. Doubtless this problem will be completely clarified in the near future.

The book consists of four chapters. The first chapter presents necessary information of the theory of elasticity of an anisotropic body. The second chapter is devoted to the presentation of two versions of the improved theory of the bending of anisotropic plates at small and large displacements. The third chapter considers the numerous problems of the determination of stresses and shears of different types of anisotropic plates under different types of loading. The fourth chapter presents the problems of stability and vibration of plates. Free and forced vibrations, static and dynamic stability, and the supersonic flutter of anisotropic and transversely isotropic plates are also considered in this chapter.

The problems presented in this book are of interest to designers of all types of machines, aircraft, ships, and rockets and are of importance in many other branches of current technology. The book could be used as a valuable information source for engineers, designers, scientists, graduate students of higher educational institutions and for other specialists who encounter the theoretical and applied problems of anisotropic plates in their work.

The author is very grateful to A. L. Gol'denveizer, who read the manuscript very carefully and contributed many valuable suggestions, and to co-workers D. V. Peshtmaldzhyan, G. E. Bagdasaryan, V. Ts. Gnuni and A. A. Khachatryan of the Institute of Mathematics and Mechanics of the Academy of Sciences of Armenya SSR. They were all helpful during the preparation of the manuscript.

Editor's Note: To minimize errors in the reproduction of the equations in this translation, the equations from the original Russian text have been used; consequently, certain minor differences exist between the symbols in the text and in the equations. Generally such minor differences cause no problem. One possible exception is the lower case Greek theta, which is used in the two forms $\theta = \vartheta$.

CHAPTER I

BASIC EQUATIONS OF THE THEORY OF ELASTICITY OF AN ANISOTROPIC BODY

1. Preliminary Remarks

This chapter presents all the necessary information from the theory of elasticity of a solid body which will be used in the succeeding pages.

A detailed presentation of the principal equations and the theory of elasticity of the anisotropic body and also of the geometrically non-linear theory of elasticity, on which this chapter is based, is given in the works of S. G. Lekhnitskii [1] and V. V. Novozhilov [2]. These two monographs are used frequently in Chapter I of this book. As a rule they are used without additional references.

2. Stressed and Deformed State of a Solid Anisotropic Body

Let the solid body in Cartesian coordinates x, y, z undergo deformation because of some applied system of forces. Then any point M of the body with coordinates x, y, z is displaced. This displacement can be represented by the following three projections of the displacement vector along the coordinates x, y, z:

$$\left.\begin{aligned} u_x &= u_x(x, y, z), \\ u_y &= u_y(x, y, z), \\ u_z &= u_z(x, y, z). \end{aligned}\right\} \tag{2.1}$$

These points are referred to below as displacements of the point M. Displacements u_x, u_y, u_z are assumed to be positive when they are directed toward positive directions of the corresponding independent variables.

A deformed state of the solid body in the vicinity of the point M is characterized by six deformation components. Three of these components, e_x, e_y, and e_z, represent relative elongation along corresponding coordinate directions x, y, z and the remaining three, e_{xy}, e_{yz}, e_{zx}, are shear deformations which take place on the corresponding coordinate planes, z = constant, x = constant, and y = constant.

The deformation components e_x, e_y, . . . , e_{zx} are related to the displacements u_x, u_y, u_z of the point M by the equations:

$$\left.\begin{aligned} e_x &= \frac{\partial u_x}{\partial x}, & e_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \\ e_y &= \frac{\partial u_y}{\partial y}, & e_{zx} &= \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}, \\ e_z &= \frac{\partial u_z}{\partial z}, & e_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}. \end{aligned}\right\} \tag{2.2}$$

A stressed state at any point M is characterized by a stress tensor with nine components, three of which are normal stresses acting along the three mutually perpendicular directions x, y, z, and the remaining six are tangential stresses acting in three mutually perpendicular coordinate planes. The tangential stresses are symmetrical and thus the number of independent stresses is equal to six.

σ_x, σ_y, σ_z are normal stresses; their subscripts indicate the direction of the external normal to that area element to which the given normal stress is related. $\tau_{xy} = \tau_{yx}$, $\tau_{xz} = \tau_{zx}$, $\tau_{yz} = \tau_{zy}$ are tangential stresses. The first subscript indicates which direction a given tangential stress acts, and the second subscript indicates the direction of the external normal to that plane on which the given stress acts.

All stresses, both normal and tangential, are positive provided they act along positive directions of the corresponding external normals while being applied to an area element with positive external normals (Figure 1).

For a given solid body in equilibrium, the equilibrium conditions in Cartesian coordinates are represented by the following three differential equations:

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + P_x &= 0, \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + P_y &= 0, \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + P_z &= 0, \end{aligned}\right\} \tag{2.3}$$

where $P_x = P_x(x, y, z)$, $P_y = P_y(x, y, z)$, $P_z = P_z(x, y, z)$ are projections of the body force (as related to a unit volume) along the directions x, y, z.

A differential equation of motion of the solid body in the Cartesian system of coordinates x, y, z is obtained when inertial terms

$$\rho \frac{\partial^2 u_x}{\partial t^2}, \quad \rho \frac{\partial^2 u_y}{\partial t^2}, \quad \rho \frac{\partial^2 u_z}{\partial t^2}$$

(ρ is the material density; t is the time) are added into the right hand side of (2.3).

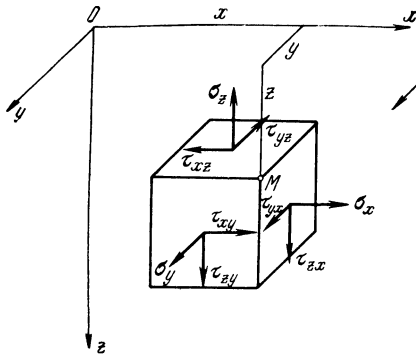

Figure 1.

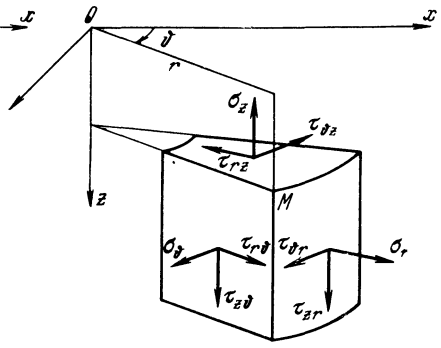

Figure 2.

Let the solid body be referred to a cylindrical coordinate system r, θ, z, with the z axis coinciding with the z axis of the Cartesian coordinate system, the radial coordinate being measured within the plane z = constant from the origin O of the Cartesian system of coordinates, and the θ angle being measured from the x axis which is assumed to be a polar axis (Figure 2).

Displacements of any point M(r, θ z) are presented in the following way in the cylindrical coordinate system:

$$\left.\begin{aligned} u_r &= u_r(r, \vartheta, z), \\ u_\vartheta &= u_\vartheta(r, \vartheta, z), \\ u_z &= u_z(r, \vartheta, z), \end{aligned}\right\} \tag{2.4}$$

where u_r is the displacement in the direction of the radial coordinate r, $u(\theta)$ is the displacement which is normal to the coordinate plane Orz, i.e., in the direction of the coordinate θ; u_z is the displacement in the direction of the coordinate z.

In the cylindrical coordinate system the deformation components e_r, e_θ, e_z represent relative elongation for coordinate directions r, θ, z, and $e_{\theta z}$, e_{rz}, $e_{r\theta}$ represent shear deformations within coordinate planes r = constant (within a plane tangent to the coordinate surface r = constant), θ = constant and z = constant.

The deformation components e_r, e_θ, . . . , $e_{r\theta}$ are related to displacements u_r, u_θ, u_z by the equations:

$$\left.\begin{aligned} e_r &= \frac{\partial u_r}{\partial r}, & e_{\vartheta z} &= \frac{\partial u_\vartheta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \vartheta}, \\ e_\vartheta &= \frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta} + \frac{u_r}{r}, & e_{rz} &= \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}, \\ e_z &= \frac{\partial u_z}{\partial z}, & e_{r\vartheta} &= \frac{1}{r}\frac{\partial u_r}{\partial \vartheta} + \frac{\partial u_\vartheta}{\partial r} - \frac{u_\vartheta}{r}. \end{aligned}\right\} \tag{2.5}$$

A stressed state in the cylindrical system of coordinates is described by a stress tensor with components σ_r, $\tau_{\theta r}$, τ_{zr}; σ_θ, $\tau_{r\theta}$, $\tau_{z\theta}$; σ_z, τ_{rz}, $\tau_{\theta z}$ acting on area elements normal to the coordinate directions r, θ, z respectively. The symmetry property is expressed here by $\tau_{\theta r} = \tau_{r\theta}$, $\tau_{zr} = \tau_{rz}$, $\tau_{z\theta} = \tau_{\theta z}$.

The equilibrium equations in this case are represented as

$$\left.\begin{aligned} \frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\vartheta}}{\partial \vartheta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\vartheta}{r} + R = 0, \\ \frac{\partial \tau_{r\vartheta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\vartheta}{\partial \vartheta} + \frac{\partial \tau_{\vartheta z}}{\partial z} + \frac{2\tau_{r\vartheta}}{r} + \Theta = 0, \\ \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\vartheta z}}{\partial \vartheta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + Z = 0, \end{aligned}\right\} \tag{2.6}$$

where $R = R(r, \theta, z)$, $\Theta = \Theta(r, \theta, z)$, $Z = Z(r, \theta, z)$ are projections of body force as referred to a unit volume in the coordiante directions r, θ, z.

When the right hand sides of equations (2.6) are replaced with inertia terms

$$\rho \frac{\partial^2 u_r}{\partial t^2}, \quad \rho \frac{\partial^2 u_\vartheta}{\partial t^2}, \quad \rho \frac{\partial^2 u_z}{\partial t^2},$$

the differential equations of motion of the solid medium in the cylindrical system of coordinates r, θ, z are obtained.

3. Finite Deformations[3]

The non-linear theory of elasticity, or as it is often called the theory of finite deformations, differs significantly from the classical (linear) theory of elasticity due to certain geometric properties contained within it.

The basic difference is that the theory of finite deformations takes into account the difference between the geometry of the deformed and the undeformed states.

It is known that the deformed state may be described by different methods in the theory of finite deformations. The deformed state here is described with the assumption that the accepted system of coordinates is material and travels with the body as it undergoes deformation. The (numerical) coordinates of points under consideration in the undeformed system coincide with coordinates of the deformed system. For example, if in the system of Cartesian coordinates for the undeformed state any point M has coordinates x, y, z, then the system of coordinates in the deformed state becomes curvilinear, generally speaking. However, the curvilinear coordinates of the point M remain unchanged, i.e., $\tilde{x} = x$, $\tilde{y} = y$, $\tilde{z} = z$ (sign $\sim$ indicates the deformed state). This descriptive method of the deformed state is known as the Lagrangian method.

The tensor components of finite deformations characterize the deformations in the theory of finite deformations. In the case when the undeformed state of a

body is described in the Cartesian coordinate system these components are:

$$\left.\begin{aligned} e_x &= \frac{\partial u_x}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2\right], \\ e_y &= \frac{\partial u_y}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial y}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2\right], \\ e_z &= \frac{\partial u_z}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial u_y}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right], \end{aligned}\right\} \tag{3.1}$$

$$\left.\begin{aligned} e_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial z}, \\ e_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} + \frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial y}\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\frac{\partial u_z}{\partial z}, \\ e_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial y}, \end{aligned}\right\} \tag{3.2}$$

where u_i are the vector components of the entire displacement; i = x, y, z. Using expressions (3.1), (3.2) and the following formulas

$$\left.\begin{aligned} E_x &= \sqrt{1+2e_x} - 1, \quad \sin\varphi_{xy} = \frac{e_{xy}}{(1+E_x)(1+E_y)}, \\ E_y &= \sqrt{1+2e_y} - 1, \quad \sin\varphi_{xz} = \frac{e_{xz}}{(1+E_x)(1+E_z)}, \\ E_z &= \sqrt{1+2e_z} - 1, \quad \sin\varphi_{yz} = \frac{e_{yz}}{(1+E_y)(1+E_z)} \end{aligned}\right\} \tag{3.3}$$

it is easy to determine the relative elongations E_x, E_y, E_z of fibers which are parallel to the x, y, z axes prior to deformations, respectively, as well as parameters φ_{xy}, φ_{xz}, φ_{yz}, which show changes in the initially right angles between fibers, and which are parallel to axes x and y, x and z, y and z before deformation.

It may be said then that the deformation components e_x, e_y, e_z characterize the elongation of those linear elements which are parallel to coordinate axes before deformation; e_{xy}, e_{xz}, e_{yz} characterize shearing displacements.

Formulas (3.3) show that the components of the deformation tensor e_x, e_y, e_z can be identified with corresponding elongations and the components e_{xy}, e_{xz}, e_{yz} can be compared with angular displacements for the case of small deformations, i.e., small elongations and displacements (they are disregarded as compared with unity).

The equilibrium equations of an infinitely small volumetric element with negligibly small deformations, when only its rotations are considered, are presented in the following manner in the initial Cartesian system of coordinates:

$$\frac{\partial}{\partial x}\left[\left(1+\frac{\partial u_x}{\partial x}\right)\sigma_x+\frac{\partial u_x}{\partial y}\tau_{xy}+\frac{\partial u_x}{\partial z}\tau_{xz}\right]+$$
$$+\frac{\partial}{\partial y}\left[\left(1+\frac{\partial u_x}{\partial x}\right)\tau_{xy}+\frac{\partial u_x}{\partial y}\sigma_y+\frac{\partial u_x}{\partial z}\tau_{yz}\right]+$$
$$+\frac{\partial}{\partial z}\left[\left(1+\frac{\partial u_x}{\partial x}\right)\tau_{xz}+\frac{\partial u_x}{\partial y}\tau_{yz}+\frac{\partial u_x}{\partial z}\sigma_z\right]+P_x=0,$$
$$\frac{\partial}{\partial x}\left[\frac{\partial u_y}{\partial x}\sigma_x+\left(1+\frac{\partial u_y}{\partial y}\right)\tau_{xy}+\frac{\partial u_y}{\partial z}\tau_{xz}\right]+$$
$$+\frac{\partial}{\partial y}\left[\frac{\partial u_y}{\partial x}\tau_{xy}+\left(1+\frac{\partial u_y}{\partial y}\right)\sigma_y+\frac{\partial u_y}{\partial z}\tau_{yz}\right]+$$
$$+\frac{\partial}{\partial z}\left[\frac{\partial u_y}{\partial x}\tau_{xz}+\left(1+\frac{\partial u_y}{\partial y}\right)\tau_{yz}+\frac{\partial u_y}{\partial z}\sigma_z\right]+P_y=0,$$
$$\frac{\partial}{\partial x}\left[\frac{\partial u_z}{\partial x}\sigma_x+\frac{\partial u_z}{\partial y}\tau_{xy}+\left(1+\frac{\partial u_z}{\partial z}\right)\tau_{xz}\right]+$$
$$+\frac{\partial}{\partial y}\left[\frac{\partial u_z}{\partial x}\tau_{xy}+\frac{\partial u_z}{\partial y}\sigma_y+\left(1+\frac{\partial u_z}{\partial z}\right)\tau_{yz}\right]+$$
$$+\frac{\partial}{\partial z}\left[\frac{\partial u_z}{\partial x}\tau_{xz}+\frac{\partial u_z}{\partial y}\tau_{yz}+\left(1+\frac{\partial u_z}{\partial z}\right)\sigma_z\right]+P_z=0; \quad (3.4)$$

where $\sigma_x, \ldots, \tau_{yz}$ are the stress components along the directions of the local axes of the curvilinear system of coordinates, which in the case of small deformations form the axes of the Cartesian system rotated with respect to the initial axes x, y, z due to deformation; $P_i (i = x, y, z)$ are the components of body forces along directions of the initial coordinate axes x, y, z.

Let the solid body be related to the cylindrical coordinate system r, θ, z (see Section 2) before deformation. Then the deformation components are:

$$e_r=\frac{\partial u_r}{\partial r}+\frac{1}{2}\left[\left(\frac{\partial u_r}{\partial r}\right)^2+\left(\frac{\partial u_\theta}{\partial r}\right)^2+\left(\frac{\partial u_z}{\partial r}\right)^2\right],$$
$$e_\theta=\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{u_r}{r}+\frac{1}{2r^2}\left[\left(\frac{\partial u_\theta}{\partial \theta}+u_r\right)^2+\right.$$
$$\left.+\left(\frac{\partial u_r}{\partial \theta}-u_\theta\right)^2+\left(\frac{\partial u_z}{\partial \theta}\right)^2\right],$$
$$e_z=\frac{\partial u_z}{\partial z}+\frac{1}{2}\left[\left(\frac{\partial u_z}{\partial z}\right)^2+\left(\frac{\partial u_z}{\partial z}\right)^2+\left(\frac{\partial u_\theta}{\partial z}\right)^2\right],$$
$$e_{r\theta}=r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)+\frac{1}{r}\frac{\partial u_r}{\partial \theta}+\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta}-u_\theta\right)\frac{\partial u_r}{\partial r}+$$

(Continued)

$$+\frac{1}{r}\left(\frac{\partial u_\vartheta}{\partial \vartheta}+u_r\right)\frac{\partial u_\vartheta}{\partial r}+\frac{1}{r}\frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial \vartheta},$$

$$e_{\vartheta z}=\frac{1}{r}\frac{\partial u_z}{\partial \vartheta}+\frac{\partial u_\vartheta}{\partial z}+\frac{1}{r}\left(\frac{\partial u_\vartheta}{\partial \vartheta}+u_r\right)\frac{\partial u_\vartheta}{\partial z}+$$

$$+\frac{1}{r}\left(\frac{\partial u_r}{\partial \vartheta}-u_\vartheta\right)\frac{\partial u_r}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \vartheta}\frac{\partial u_z}{\partial z},$$

$$e_{rz}=\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial r}+\frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial z}+\frac{\partial u_\vartheta}{\partial r}\frac{\partial u_\vartheta}{\partial z}+\frac{\partial u_z}{\partial z}\frac{\partial u_z}{\partial r},$$

(3.5)

where, as usual, u_i ($i = r, \theta, z$) are the vector components of the displacement in the direction of r, θ z.

The equilibrium equations in the initial cylindrical system of coordinates are:

$$\left.\begin{aligned}
&\frac{\partial}{\partial r}r\left[\left(1+\frac{\partial u_r}{\partial r}\right)\sigma_r+\frac{1}{r}\left(\frac{\partial u_r}{\partial \vartheta}-u_\vartheta\right)\tau_{r\vartheta}+\frac{\partial u_r}{\partial z}\tau_{rz}\right]+\\
&+\frac{\partial}{\partial \vartheta}\left[\left(1+\frac{\partial u_r}{\partial r}\right)\tau_{\vartheta r}+\frac{1}{r}\left(\frac{\partial u_r}{\partial \vartheta}-u_\vartheta\right)\sigma_\vartheta+\frac{\partial u_r}{\partial z}\tau_{\vartheta z}\right]+\\
&+r\frac{\partial}{\partial z}\left[\left(1+\frac{\partial u_r}{\partial r}\right)\tau_{zr}+\frac{1}{r}\left(\frac{\partial u_r}{\partial \vartheta}-u_\vartheta\right)\tau_{z\vartheta}+\frac{\partial u_r}{\partial z}\sigma\right]-\\
&-\frac{\partial u_\vartheta}{\partial r}\tau_{\vartheta r}-\left(1+\frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta}+\frac{u_r}{r}\right)\sigma_\vartheta-\frac{\partial u_\vartheta}{\partial z}\tau_{\vartheta z}+rR=0,\\
&\frac{\partial}{\partial r}r\left[\frac{\partial u_\vartheta}{\partial r}\sigma_r+\left(1+\frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta}+\frac{u_r}{r}\right)\tau_{r\vartheta}+\frac{\partial u_\vartheta}{\partial z}\tau_{rz}\right]+\\
&+\frac{\partial}{\partial \vartheta}\left[\frac{\partial u_\vartheta}{\partial r}\tau_{\vartheta r}+\left(1+\frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta}+\frac{u_r}{r}\right)\sigma_\vartheta+\frac{\partial u_\vartheta}{\partial z}\tau_{\vartheta z}\right]+\\
&+r\frac{\partial}{\partial z}\left[\frac{\partial u_\vartheta}{\partial r}\tau_{zr}+\left(1+\frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta}+\frac{u_r}{r}\right)\tau_{z\vartheta}+\frac{\partial u_\vartheta}{\partial z}\sigma_z\right]+\\
&+\left(1+\frac{\partial u_r}{\partial r}\right)\tau_{\vartheta r}+\frac{1}{r}\left(\frac{\partial u_r}{\partial \vartheta}-u_\vartheta\right)\sigma_\vartheta+\frac{\partial u_r}{\partial z}\tau_{\vartheta z}+r\Theta=0,\\
&\frac{\partial}{\partial r}r\left[\frac{\partial u_z}{\partial r}\sigma_r+\frac{1}{r}\frac{\partial u_z}{\partial \vartheta}\tau_{r\vartheta}+\left(1+\frac{\partial u_z}{\partial z}\right)\tau_{rz}\right]+\\
&+\frac{\partial}{\partial \vartheta}\left[\frac{\partial u_z}{\partial r}\tau_{\vartheta r}+\frac{1}{r}\frac{\partial u_z}{\partial \vartheta}\sigma_\vartheta+\left(1+\frac{\partial u_z}{\partial z}\right)\tau_{rz}\right]+\\
&+r\frac{\partial}{\partial z}\left[\frac{\partial u_z}{\partial r}\tau_{zr}+\frac{1}{r}\frac{\partial u_z}{\partial \vartheta}\tau_{z\vartheta}+\left(1+\frac{\partial u_z}{\partial z}\right)\sigma_z\right]+rZ=0,
\end{aligned}\right\}\quad(3.6)$$

In this case $\sigma_r, \ldots, \tau_{\theta z}$ are the stress components along the directions of the local axes of the curvilinear system of coordinates, which in the case of small deformations form the axes of the cylindrical coordinate system, which is rotated with respect to the initial cylindrical coordinate system r, θ, z according to the rotation obtained as a result of deformation; R, Θ, Z are the components of body forces along directions r, θ, z of the initial coordinate system.

By disregarding all non-linear terms in (3.1) – (3.6) it is easy to obtain from them the corresponding equations of the linear theory, i.e., equations (2.2) – (2.6).

When corresponding inertial terms are added to the equilibrium equations (3.4) and (3.6) the equations of motion of the solid medium in the case of finite deformations are obtained.

4. Generalized Hooke's Law [4, 5]

This book concerns only those deformations for which the generalized Hooke's law is valid.

Equations for the generalized Hooke's law for a uniform elastic body can be expressed in the following way in an arbitrary orthogonal system of coordinates x, y, z:

$$\left.\begin{aligned} e_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{14}\tau_{yz} + a_{15}\tau_{xz} + a_{16}\tau_{xy}, \\ e_y &= a_{12}\sigma_x + a_{22}\sigma_y + \ldots\ldots\ldots + a_{26}\tau_{xy}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ e_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + \ldots\ldots\ldots + a_{66}\tau_{xy}, \end{aligned}\right\} \quad (4.1)$$

where a_{ik} are the elastic constants (deformation coefficients). In the general case the number of independent elastic constants is equal to 21 because $a_{ik} = a_{ki}$.

The elastic potential for a unit volume element is:

$$\begin{aligned} \bar{V} = {} & \frac{1}{2}a_{11}\sigma_x^2 + \sigma_x(a_{12}\sigma_y + a_{13}\sigma_z + a_{14}\tau_{yz} + a_{15}\tau_{zx} + a_{16}\tau_{xy}) + \\ & + \frac{1}{2}a_{22}\sigma_y^2 + \sigma_y(a_{23}\sigma_z + a_{24}\tau_{yz} + a_{25}\tau_{zx} + a_{26}\tau_{xy}) + \\ & + \frac{1}{2}a_{33}\sigma_z^2 + \sigma_z(a_{34}\tau_{yz} + a_{35}\tau_{zx} + a_{36}\tau_{xy}) + \\ & + \frac{1}{2}a_{44}\tau_{yz}^2 + \tau_{yz}(a_{45}\tau_{zx} + a_{46}\tau_{xy}) + \frac{1}{2}a_{55}\tau_{zx}^2 + \tau_{zx}a_{56}\tau_{xy} + \frac{1}{2}a_{66}\tau_{xy}^2. \end{aligned} \quad (4.2)$$

On the basis of (4.1) $\bar{V}$ can be represented in the following form:

$$\bar{V} = \frac{1}{2}(\sigma_x e_x + \sigma_y e_y + \sigma_z e_z + \tau_{yz}e_{yz} + \tau_{zx}e_{zx} + \tau_{xy}e_{xy}). \quad (4.3)$$

The strain energy of a whole body is determined by integrating $\bar{V}$ over the entire body volume:

$$V = \int\int\int_{\omega} \overline{V}\, d\omega. \tag{4.4}$$

If there is symmetry in the internal structure of the anisotropic body material, then certain elastic symmetry is also present in its elastic properties, i.e., there are symmetric directions in the body with respect to which the elastic properties of the material are identical. When the anisotropic body possesses such elastic symmetry the equations of the generalized Hooke's law are simplified. Below, some important cases of such elastic symmetry are described.

1. One elastic symmetry plane. Let each point of the body have a plane with any two directions symmetrical with respect to it which are equivalent in their elastic properties. The equations of the generalized Hooke's law are then presented as follows, on the assumption that the plane of elastic symmetry at each point of the body is parallel to the coordinate plane Oxy:

$$\left.\begin{aligned} e_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z + a_{16}\tau_{xy},\\ e_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z + a_{26}\tau_{xy},\\ e_z &= a_{13}\sigma_x + a_{23}\sigma_y + a_{33}\sigma_z + a_{36}\tau_{xy},\\ e_{yz} &= a_{44}\tau_{yz} + a_{45}\tau_{xz},\\ e_{xz} &= a_{45}\tau_{yz} + a_{55}\tau_{xz},\\ e_{xy} &= a_{16}\sigma_x + a_{26}\sigma_y + a_{36}\sigma_z + a_{66}\tau_{xy}. \end{aligned}\right\} \tag{4.5}$$

The number of independent elastic constants a_{ik} in this case is reduced to thirteen.

Equations (4.5) can be represented in terms of engineering constants:

$$\begin{aligned} e_x &= \frac{1}{E_1}\sigma_x - \frac{\nu_{12}}{E_2}\sigma_y - \frac{\nu_{13}}{E_3}\sigma_z + \frac{\eta_{1,12}}{G_{12}}\tau_{xy},\\ e_y &= -\frac{\nu_{21}}{E_1}\sigma_x + \frac{1}{E_2}\sigma_y - \frac{\nu_{23}}{E_3}\sigma_z + \frac{\eta_{2,12}}{G_{12}}\tau_{xy},\\ e_z &= -\frac{\nu_{31}}{E_1}\sigma_x - \frac{\nu_{32}}{E_2}\sigma_y + \frac{1}{E_3}\sigma_z + \frac{\eta_{3,12}}{G_{12}}\tau_{xy},\\ e_{yz} &= \frac{1}{G_{23}}\tau_{yz} + \frac{\mu_{23,31}}{G_{31}}\tau_{xz},\\ e_{xz} &= \frac{\mu_{31,23}}{G_{23}}\tau_{yz} + \frac{1}{G_{31}}\tau_{xz},\\ e_{xy} &= \frac{\eta_{12,1}}{E_1}\sigma_x + \frac{\eta_{12,2}}{E_2}\sigma_y + \frac{\eta_{12,3}}{E_3}\sigma_x + \frac{1}{G_{12}}\tau_{xy}. \end{aligned} \tag{4.6}$$

Here $E_1 = E_x$, $E_2 = E_y$, $E_3 = E_z$ are the Young's moduli along principal elasticity directions x, y, z, respectively; $G_{23} = G_{yz}$, $G_{13} = G_{xz}$, $G_{12} = G_{xy}$ are shear moduli which characterize angular changes between principal directions y and z, x and z, x and y; $\nu_{12} = \nu_{xy}$, $\nu_{21} = \nu_{yx}$, $\nu_{13} = \nu_{xz}$, $\nu_{31} = \nu_{zx}$, $\nu_{23} = \nu_{yz}$, $\nu_{32} = \nu_{zy}$ are the Poisson's ratios which characterize the transverse contraction (expansion) during tension (compression) in the direction of coordinate axes (the first index shows the direction of contraction or expansion, the second index indicates the direction of the acting force); $\mu_{23,31} = \mu_{yz,zx}$, $\mu_{31,23} = \mu_{zx,yz}$ are Chentsov coefficients which characterize the shear strains within planes parallel to the first referenced coordinate plane which are produced by shear stress parallel to another coordinate plane (for example, $\mu_{zx,yz}$ characterizes the shear strain within the plane parallel to Ozx caused by the stress τ_{yz}); $\eta_{12,1} = \eta_{xy,x}$, $\eta_{12,2} = \eta_{xy,y}$, $\eta_{12,3} = \eta_{xy,z}$ are the coefficients of the reciprocal effect of the first kind which characterize shear strains within planes parallel to the first referenced coordinate plane due to normal stresses (for example, $\eta_{xy,x}$ characterizes a shear strain within the plane parallel to Oxy which was produced by the stress σ_x); $\eta_{1,12} = \eta_{x,xy}$, $\eta_{2,12} = \eta_{y,xy}$, $\eta_{3,12} = \eta_{z,xy}$ are coefficients of the reciprocal effect of the second kind which characterize elongations in directions parallel to the first referenced coordinate axis and which are produced by shear stresses. (For example, $\eta_{z,xy}$ characterizes elongation in the Oz direction caused by the stress τ_{xy}).

2. Three planes of elastic symmetry. Let three mutually perpendicular planes of elastic symmetry pass through each point of a body. By assuming that these planes are perpendicular to corresponding coordinate axes x, y, z at each point the following equations of the generalized Hooke's law are obtained:

$$\left.\begin{aligned} e_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z, & e_{yz} &= a_{44}\tau_{yz}, \\ e_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{23}\sigma_z, & e_{zx} &= a_{55}\tau_{zx}, \\ e_z &= a_{13}\sigma_x + a_{23}\sigma_y + a_{33}\sigma_z, & e_{xy} &= a_{66}\tau_{xy}. \end{aligned}\right\} \tag{4.7}$$

The number of independent elastic constants a_{ik} is nine in this case.

Equations (4.7) can also be represented in terms of engineering constants:

$$\left.\begin{aligned} e_x &= \frac{1}{E_1}\sigma_x - \frac{\nu_{12}}{E_2}\sigma_y - \frac{\nu_{13}}{E_3}\sigma_z, & e_{yz} &= \frac{1}{G_{23}}\tau_{yz} \\ e_y &= \frac{1}{E_2}\sigma_y - \frac{\nu_{21}}{E_1}\sigma_x - \frac{\nu_{23}}{E_3}\sigma_z, & e_{zx} &= \frac{1}{G_{13}}\tau_{zx}, \\ e_z &= \frac{1}{E_3}\sigma_z - \frac{\nu_{31}}{E_1}\sigma_x - \frac{\nu_{32}}{E_2}\sigma_y, & e_{xy} &= \frac{1}{G_{12}}\tau_{xy} \end{aligned}\right\} \tag{4.8}$$

The following equalities exist because of the symmetry of equations (4.7).

$$E_2\nu_{21} = E_1\nu_{12}, \quad E_3\nu_{32} = E_2\nu_{23}, \quad E_1\nu_{13} = E_3\nu_{31}. \tag{4.9}$$

A body which has three mutually perpendicular planes of elastic symmetry at each point is called the orthotropically anisotropic or orthotropic body.

3. A plane of isotropy. If a plane in which all directions are equivalent with respect to elastic properties passes through each point of the body, then the equations of the generalized Hooke's law for a system of coordinates with the z-axis normal to this plane are:

$$\left.\begin{aligned} e_x &= a_{11}\sigma_x + a_{12}\sigma_y + a_{13}\sigma_z, & e_{yz} &= a_{44}\tau_{yz}, \\ e_y &= a_{12}\sigma_x + a_{22}\sigma_y + a_{13}\sigma_z, & e_{xz} &= a_{44}\tau_{xz}, \\ e_z &= a_{13}(\sigma_x + \sigma_y) + a_{33}\sigma_z, & e_{xy} &= 2(a_{11} - a_{12})\tau_{xy}. \end{aligned}\right\} \tag{4.10}$$

The number of independent elastic constants in this case is five.

Equations (4.10) can also be represented in terms of engineering constants as follows:

$$\left.\begin{aligned} e_x &= \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y - \frac{\nu'}{E'}\sigma_z, & e_{yz} &= \frac{1}{G'}\tau_{yz}, \\ e_y &= -\frac{\nu}{E}\sigma_x + \frac{1}{E}\sigma_y - \frac{\nu'}{E'}\sigma_z, & e_{xz} &= \frac{1}{G'}\tau_{xz}, \\ e_z &= -\frac{\nu''}{E}\sigma_x - \frac{\nu''}{E}\sigma_y + \frac{1}{E'}\sigma_z, & e_{xy} &= \frac{1}{G}\tau_{xy}. \end{aligned}\right\} \tag{4.11}$$

The following equalities exist because of the symmetry of (4.10):

$$\nu''E' = \nu'E. \tag{4.12}$$

In equations (4.11) and (4.12) E designates the Young's modulus for directions within the isotropy plane; G = E/2(1 + ν) is the shear modulus for the isotropy plane; E′ is the Young's modulus for the direction perpendicular to the plane of isotropy; G′ is the shear modulus for planes normal to the plane of isotropy; ν is the Poisson's coefficient which characterizes the contraction within the plane of isotropy for forces applied within the same plane; ν' is the Poisson's coefficient which characterizes contraction within the plane of isotropy due to forces in the direction perpendicular to it; ν'' is the Poisson's coefficient which characterizes contraction in the direction perpendicular to the plane of isotropy due to forces within the plane of isotropy.

The body with the above elastic properties is called transversely isotropic.

In this case of elastic symmetry the direction perpendicular to the isotropy plane and all other directions within this plane are principal directions.

4. Complete symmetry, the isotropic body. All directions in an isotropic body are equivalent and any plane passing through any body point is a plane of elastic symmetry. The equations of the generalized Hooke's law are:

$$\left.\begin{aligned} e_x &= \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)], \quad e_{yz} = \frac{1}{G}\tau_{yz}, \\ e_y &= \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)], \quad e_{zx} = \frac{1}{G}\tau_{zx}, \\ e_z &= \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)], \quad e_{xy} = \frac{1}{G}\tau_{xy}. \end{aligned}\right\} \tag{4.13}$$

Here E is Young's modulus; ν is the Poisson's coefficient; G = E/2 (1 + ν) is the shear modulus. There are two independent elastic constants.

Cylindrically anisotropic bodies are also of interest and are discussed in this book in addition to rectilinearly anisotropic bodies.

A body with cylindrical anisotropy possesses the following elastic properties. A straight line γ which is the axis of anisotropy is always referred to for the case of cylindrical anisotropy (this line may be either inside or outside of the body). All directions which cross the axis of anisotropy at right angles are equivalent; all directions which are parallel to the axis of anisotropy and all directions which are orthogonal to the first two are correspondingly equivalent.

Equations of the generalized Hooke's law for a body with cylindrical anisotropy of the general type, in the cylindrical coordinate system r, θ, z, with the z axis coinciding with the axis of anisotropy, are presented in the form:

$$\left.\begin{aligned} e_r &= a_{11}\sigma_r + a_{12}\sigma_\theta + a_{13}\sigma_z + a_{14}\tau_{\theta z} + a_{15}\tau_{rz} + a_{16}\tau_{r\theta}, \\ e_\theta &= a_{12}\sigma_r + a_{22}\sigma_\theta + \ldots\ldots\ldots + a_{26}\tau_{r\theta}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ e_{r\theta} &= a_{16}\sigma_r + a_{26}\sigma_\theta + \ldots\ldots\ldots + a_{66}\tau_{r\theta}. \end{aligned}\right\} \tag{4.14}$$

The number of independent elastic constants is 21.

Special cases of anisotropy with different types of elastic symmetry are also possible in the case of cylindrical anisotropy. For example, if three planes of elastic symmetry are present at each point of the body, then one of them is normal to the axis of anisotropy, the second passes through it, and the third is orthogonal to the first two. In this case

equations (4.14) will take the form of equations (4.7) (with ~~different~~ indexes for stresses). The body is called orthotropic with cylindrical anisotropy in this case.

Equations of the type (4.7) for the case of cylindrical anisotropy can also be expressed in terms of engineering constants:

$$\left.\begin{aligned} e_r &= \frac{1}{E_r}\sigma_r - \frac{\nu_{r\vartheta}}{E_\vartheta}\sigma_\vartheta - \frac{\nu_{rz}}{E_z}\sigma_z, & e_{\vartheta z} &= \frac{1}{G_{\vartheta z}}\tau_{\vartheta z}, \\ e_\vartheta &= -\frac{\nu_{\vartheta r}}{E_r}\sigma_r + \frac{1}{E_\vartheta}\sigma_\vartheta - \frac{\nu_{\vartheta z}}{E_z}\sigma_z, & e_{rz} &= \frac{1}{G_{rz}}\tau_{rz}, \\ e_z &= -\frac{\nu_{zr}}{E_r}\sigma_r - \frac{\nu_{z\vartheta}}{E_\vartheta}\sigma_\vartheta + \frac{1}{E_z}\sigma_z, & e_{r\vartheta} &= \frac{1}{G_{r\vartheta}}\tau_{r\vartheta}, \end{aligned}\right\} \qquad (4.15)$$

where E_r, E_θ, E_z are Young's moduli for the principal elasticity directions r, θ, z, respectively; $G_{\theta z}$, G_{rz}, $G_{r\theta}$ are shear moduli which characterize the angular changes between principal directions θ and z, r and z, r and θ; $\nu_{r\theta}$, $\nu_{\theta r}$, ν_{rz}, ν_{zr}, $\nu_{\theta z}$, $\nu_{z\theta}$ are Poisson's coefficients (the first index shows the contraction (expansion) direction, the second index shows the direction of the acting force).

Equalities analogous to (3.9) also exist here because of the symmetry:

$$E_\vartheta \nu_{\vartheta r} = E_r \nu_{r\vartheta}, \quad E_z \nu_{zr} = E_r \nu_{rz}, \quad E_z \nu_{z\vartheta} = E_\vartheta \nu_{\vartheta z}. \qquad (4.16)$$

In this case the number of independent elastic constants is nine.

Equations of the generalized Hooke's law in the cylindrical coordinate system can also be presented in a similar way for other cases of elastic symmetry.

The equations of the generalized Hooke's law are also valid for finite deformations in the case of small elongations and shearing strains in comparison with unity. The equations of the generalized Hooke's law describe in this case the linear relation between components of the deformation tensor e_{ik} and stress components according to the directions of the local axes of the curvilinear system of coordinates. This relation does not depend explicitly on Lagrange coordinates. The body is uniform with respect to curvilinear coordinates, i.e., the body under consideration is curvilinearly anisotropic. This means that the mechanical properties of the body material in any local system (axes) of curvilinear coordinates is described identically. The principal anisotropy axes change directions from point to point, in accordance to direction changes of axes of the local system.[6]

5. Transformation of Elastic Constants Due to Rotation of Coordinates [7]

In order to solve certain problems in the theory of elasticity of an anisotropic

body, it is necessary to know the values of elastic constants for some system of coordinates x′, y′, z′, when the elastic constants in another system of coordinates x, y, z are known.

Consider now a generalized state of plane stress of an anisotropic plate. Let the plate material have only one plane of elastic symmetry at each point which is parallel to the coordinate planes Oxy and Ox′y′. Let the coordinate systems (x, y, z) and (x′, y′, z′) be produced from each other by rotating the system at some angle φ around the common z-axis.

For this case the following transformation formulas for the elastic constants are obtained:

$$a'_{11} = a_{11}\cos^4\varphi + (2a_{12} + a_{66})\sin^2\varphi\cos^2\varphi + a_{22}\sin^4\varphi + \\ + (a_{16}\cos^2\varphi + a_{26}\sin^2\varphi)\sin 2\varphi,$$

$$a'_{22} = a_{11}\sin^4\varphi + (2a_{12} + a_{66})\sin^2\varphi\cos^2\varphi + a_{22}\cos^4\varphi - \\ - (a_{16}\sin^2\varphi + a_{26}\cos^2\varphi)\sin 2\varphi,$$

$$a'_{12} = (a_{11} + a_{22} - 2a_{12} - a_{66})\sin^2\varphi\cos^2\varphi + a_{12} + \\ + \frac{1}{2}(a_{26} - a_{16})\sin 2\varphi\cos 2\varphi,$$

$$a'_{66} = 4(a_{11} + a_{22} - 2a_{12} - a_{66})\sin^2\varphi\cos^2\varphi + a_{66} + \\ + 2(a_{26} - a_{16})\sin 2\varphi\cos 2\varphi,$$

$$a'_{16} = \left[a_{22}\sin^2\varphi - a_{11}\cos^2\varphi + \frac{1}{2}(2a_{12} + a_{66})\cos 2\varphi\right]\sin 2\varphi + \\ + a_{16}\cos^2\varphi(\cos^2\varphi - 3\sin^2\varphi) + a_{26}\sin^2\varphi(3\cos^2\varphi - \sin^2\varphi),$$

$$a'_{26} = \left[a_{22}\cos^2\varphi - a_{11}\sin^2\varphi - \frac{1}{2}(2a_{12} + a_{66})\cos 2\varphi\right]\sin 2\varphi + \\ + a_{16}\sin^2\varphi(3\cos^2\varphi - \sin^2\varphi) + a_{26}\cos^2\varphi(\cos^2\varphi - 3\sin^2\varphi).$$

The following invariant relationships should also be mentioned:

$$a'_{11} + a'_{22} + 2a'_{12} = a_{11} + a_{22} + 2a_{12}, \quad a'_{66} - 4a'_{12} = a_{66} - 4a_{12}.$$

In the particular case of an orthotropic plate with principal directions of elasticity which coincide with the directions of the coordinate axes x and y and with the principal elastic constants which are known, it is easy to find the elastic constants of the rotated system of coordinates x′y′z. The following formulas are available to accomplish this:

$$a'_{11} = \frac{\cos^4\varphi}{E_1} + \left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)\sin^2\varphi\cos^2\varphi + \frac{\sin^4\varphi}{E_2}, \tag{5.1}$$

$$a'_{22} = \frac{\sin^4\varphi}{E_1} + \left(\frac{1}{G} - \frac{2\nu_2}{E_2}\right)\sin^2\varphi\cos^2\varphi + \frac{\cos^4\varphi}{E_2}, \tag{5.2}$$

$$a'_{12} = \left(\frac{1+\nu_1}{E_1} + \frac{1+\nu_2}{E_2} - \frac{1}{G}\right)\frac{1}{4}\sin^2 2\varphi - \frac{\nu_1}{E_1}, \tag{5.3}$$

$$a'_{66} = \left(\frac{1+\nu_1}{E_1} + \frac{1+\nu_2}{E_2} - \frac{1}{G}\right)\sin^2 2\varphi + \frac{1}{G}, \tag{5.4}$$

$$a'_{16} = \left[\frac{\sin^2\varphi}{E_2} - \frac{\cos^2\varphi}{E_1} + \frac{1}{2}\left(\frac{1}{G} - \frac{2\nu_1}{E_1}\right)\cos 2\varphi\right]\sin 2\varphi, \tag{5.5}$$

$$a'_{26} = \left[\frac{\cos^2\varphi}{E_2} - \frac{\sin^2\varphi}{E_1} - \frac{1}{2}\left(\frac{1}{G} - \frac{2\nu_2}{E_2}\right)\cos 2\varphi\right]\sin 2\varphi. \tag{5.6}$$

Three additional formulas are needed for this special case of an orthotropic plate:

$$a'_{44} = \frac{\cos^2\varphi}{G_{23}} + \frac{\sin^2\varphi}{G_{13}}, \tag{5.7}$$

$$a'_{55} = \frac{\sin^2\varphi}{G_{23}} + \frac{\cos^2\varphi}{G_{13}}, \tag{5.8}$$

$$a'_{45} = \left(\frac{1}{G_{23}} - \frac{1}{G_{13}}\right)\sin\varphi\cos\varphi. \tag{5.9}$$

There are also the following invariant relationships:

$$\left.\begin{aligned} \frac{1}{E_1'} + \frac{1}{E_2'} - \frac{2\nu_1'}{E_1'} &= \frac{1}{E_1} + \frac{1}{E_2} - \frac{2\nu_1}{E_1}, \\ \frac{1}{G'} + \frac{4\nu_1'}{E_1'} &= \frac{1}{G} + \frac{4\nu_1}{E_1}. \end{aligned}\right\} \tag{5.10}$$

In equations (5.1) – (5.7), the following notation has been used:

$$\nu_{12} = \nu_2, \qquad \nu_{21} = \nu_1, \qquad (E_1\nu_2 = E_2\nu_1), \qquad G_{12} = G. \tag{5.11}$$

This notation will be used for the remainder of the book.

6. Elastic Constants for Some Anisotropic Materials

Among the many types of existing anisotropic materials only some non-crystalline materials which are widely used in industry are discussed here. Included in this class of materials are natural wood, rolled metal sheets, concrete, paper, delta wood, plywood, fabrics, different structural plates, etc. As a first approximation it is assumed that the materials under consideration are uniform and orthotropic. The principal directions of elasticity coincide with the coordinate lines x, y, z.

1. Natural wood. Disregarding the nonuniformity and the curvature of annual rings, natural wood reveals three planes of elastic symmetry: one of these, Oyz, is normal to the wood fibers; the second, Oxy, is parallel to the plane of annual layers, and the third, Oxz, is orthogonal to the first two.

   The numerical values of elastic constants for pine are taken from the work by A. L. Rabinovich and are [8]:

$$E_1 = 1 \cdot 10^5 \text{ kg/cm}^2; \qquad E_2 = 0.042 \cdot 10^5 \text{ kg/cm}^2;$$
$$\nu_2 = 0.01; \qquad G_{12} = 0.075 \cdot 10^5 \text{ kg/cm}^2.$$

2. Delta wood. Delta wood is a layered wood material which is produced by a hot pressing of the wood stack consisting of a great number of layers impregnated with resin. Assuming that the x-axis coincides with the dominant direction of fibers the following numerical values for averaged elastic constants are available[9]:

$$E_1 = 3.05 \cdot 10^5 \text{ kg/cm}^2; \qquad E_2 = 0.467 \cdot 10^5 \text{ kg/cm}^2;$$
$$\nu_2 = 0.02; \qquad G_{12} = 0.22 \cdot 10^5 \text{ kg/cm}^2.$$

3. Plywood. A sheet of plywood is layered and anisotropic. However, as a first approximation it can be assumed that plywood also represents a uniform orthotropic material. By assuming that the x-axis coincides with the direction of the external fibers the following data are available for a three-layer plywood 1.0 – 5.0 mm thick[10]:

$$E_1 = 1.2 \cdot 10^5 \text{ kg/cm}^2; \qquad E_2 = 0.644 \cdot 10^5 \text{ kg/cm}^2;$$
$$\nu_2 = 0.044; \qquad G_{12} = 0.072 \cdot 10^5 \text{ kg/cm}^2.$$

4. Glass laminate. Glass laminate is an anisotropic laminated plastic which consists of glass cloth impregnated with resin. However, it is assumed as a first approximation that the glass laminate is a uniform anisotropic material. Assuming that the x-axis coincides with the direction of the glass cloth basis, the following data are available for the glass laminate (3 – 10 mm thick) made of polyester acrylate 911–MS–KhO and the glass cloth T_1 (GOST 8481–57)[11]:

$$E_1 = 1.37 \cdot 10^5 \text{ kg/cm}^2; \qquad E_2 = 0.9 \cdot 10^5 \text{ kg/cm}^2;$$
$$G_{12} = 1.22 \cdot 10^5 \text{ kg/cm}^2.$$

5. Fiber-glass reinforced plastics. Fiber-glass reinforced plastic AG–4s is an anisotropic laminated material which consists of oriented glass fibers impregnated with resin. It is usually produced in the form of bands of different widths by continuous methods. Individual items from this material are produced by extrusion.

   Assuming that the x-axis coincides with the direction of higher rigidity the data available for this material are[12]:

$$E_1 = 2.1 \cdot 10^5 \text{ kg/cm}^2; \qquad E_2 = 1.6 \cdot 10^5 \text{ kg/cm}^2;$$
$$\nu_2 = 0.07; \qquad G_{12} = 0.42 \cdot 10^5 \text{ kg/cm}^2.$$

6. SVAM. SVAM is an acronym for glass-fiber anisotropic material made of glass veneer sheets impregnated with resin. For this case we also consider that we deal with a uniform anisotropic material as a first approximation. On the assumption that the x-axis coincides with the direction of the majority of glass fibers the following data are available for SVAM (5 mm thick with a ratio of glass fibers 1:5) made of epoxy resin ED–6 and glass veneer sheets 0.35 – 0.4 mm thick[13]:

$$E_1 = 3.05 \cdot 10^5 \text{ kg/cm}^2; \qquad E_2 = 1.88 \cdot 10^5 \text{ kg/cm}^2;$$
$$\nu_2 = 0.12; \qquad G_{12} = 0.49 \cdot 10^5 \text{ kg/cm}^2.$$

7. Concrete. Concrete acquires anisotropic properties because of the method with which it is produced and cast. The numerical values of the elastic moduli of light concrete are[14]:

$$E_1 = 1.08 \cdot 10^5 \text{ kg/cm}^2; \qquad E_2 = 0.81 \cdot 10^5 \text{ kg/cm}^2.$$

8. Paper. In general all types of paper sheets are anisotropic. For example, values for the semi-Bristol paper produced by the Goznak plant with a density of 160 g/m² are[15]:

$$E_1 = 30.1 \cdot 10^3 \text{ kg/cm}^2; \qquad E_2 = 22.6 \cdot 10^3 \text{ kg/cm}^2;$$
$$\nu_2 = 0.23; \qquad G_{12} = 9.96 \cdot 10^3 \text{ kg/cm}^2.$$

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CHAPTER II

GENERAL THEORY OF ANISOTROPIC PLATES

1. Initial Principles and Assumptions of the General Theory

Consider an anisotropic plate of uniform thickness h. The plate material obeys the generalized Hooke's law (1.4.1) and has only one plane of elastic symmetry at each point which is parallel to the plate middle plane (plates with a more general anisotropy are not considered in this book). The middle plane in this case is equidistant from two parallel planes which form the upper and lower surfaces of the plate. In addition, for the case of finite plate dimensions, the plate, in the general case, is also restricted by some cylindrical surface C with components which, for the undeformed plate, are normal to the plate middle plane.

Let the plate be loaded by surface forces which cause both in-plane and bending deformations.

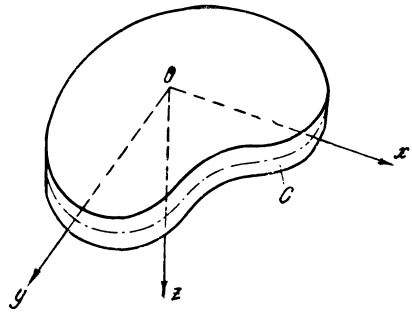

Figure 3.

On the assumption that the plate middle plane coincides with the coordinate plane Oxy and the coordinate z-axis is directed as shown in Figure 3, the generalized Hooke's law for this plate can be represented in the form (1.4.5) or (1.4.6).

The theory of anisotropic plates suggested here is based on the following assumptions [1, 2, 3, 4]:

(a) The displacement u_z which is normal to the plate middle plane does not depend on the z-coordinate;

(b) the shear stresses τ_{xz} and τ_{yz} or the corresponding deformations e_{xz} and e_{yz} change according to a given law with respect to the plate thickness.

The first assumption, in fact, coincides with the corresponding assumption of the classical theory[5], while the second is new and allows investigation of the phenomena related to transverse shear.

2. Base Equations and Identities

Considering the assumptions (a) and (b) we assume, in fact, that

$$e_z = 0, \tag{2.1}$$

$$\left.\begin{aligned} \tau_{xz} &= \frac{X^+ - X^-}{2} + \frac{z}{h}(X^+ + X^-) + f_1(z)\varphi(x,y), \\ \tau_{yz} &= \frac{Y^+ - Y^-}{2} + \frac{z}{h}(Y^+ + Y^-) + f_2(z)\psi(x,y), \end{aligned}\right\} \tag{2.2}$$

where $X^+(x, y)$, $Y^+(x, y)$, and $X^-(x, y)$, $Y^-(x, y)$, are tangential components of the force vectors applied to the external plate planes ($z = \frac{1}{2}h$, $z = -\frac{1}{2}h$); $\varphi(x, y)$ and $\psi(x, y)$ are arbitrary functions of the coordinates x and y which are to be determined; $f_i(z)$ is the function which characterizes the variation of shear stresses τ_{xz} and τ_{yz} with respect to the plate thickness. In addition $f_i(\pm h/2) = 0$ (Figure 4).

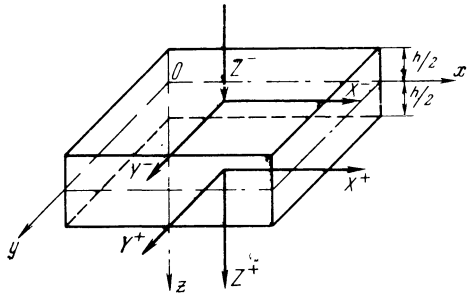

Figure 4.

It is not difficult to see from (2.2) that tangential stresses τ_{xz} and τ_{yz} satisfy the following boundary conditions of the plate:

$$\text{at} \quad z = \frac{h}{2} \quad \tau_{xz} = X^+, \tau_{yz} = Y^+$$

$$\text{at} \quad z = -\frac{h}{2} \quad \tau_{xz} = -X^-, \tau_{yz} = -Y^-$$

Solving the generalized Hooke's law equations (1.4.5) for the stresses σ_x, σ_y, τ_{xz}, τ_{yz}, τ_{xy}, one obtains:

$$\sigma_x = B_{11}e_x + B_{12}e_y + B_{16}e_{xy} - A_1\sigma_z, \tag{2.3}$$

$$\sigma_y = B_{22}e_y + B_{12}e_x + B_{26}e_{xy} - A_2\sigma_z, \tag{2.4}$$

$$\tau_{xy} = B_{66}e_{xy} + B_{16}e_x + B_{26}e_y - A_3\sigma_z, \tag{2.5}$$

$$\tau_{xz} = B_{55}e_{xz} + B_{45}e_{yz}, \tag{2.6}$$

$$\tau_{yz} = B_{44}e_{yz} + B_{45}e_{xz}, \tag{2.7}$$

where

$$\left.\begin{aligned} B_{11} &= \frac{a_{22}a_{66} - a_{26}^2}{\Omega}, & B_{22} &= \frac{a_{11}a_{66} - a_{16}^2}{\Omega} \\ B_{12} &= \frac{a_{16}a_{26} - a_{12}a_{66}}{\Omega}, & B_{66} &= \frac{a_{11}a_{22} - a_{12}^2}{\Omega} \\ B_{16} &= \frac{a_{12}a_{26} - a_{22}a_{16}}{\Omega}, & B_{26} &= \frac{a_{12}a_{16} - a_{11}a_{26}}{\Omega} \end{aligned}\right\} \tag{2.8}$$

$$\Omega = (a_{11}a_{22} - a_{12}^2)a_{66} + 2a_{12}a_{16}a_{26} - a_{11}a_{26}^2 - a_{22}a_{16}^2, \tag{2.9}$$

$$\left.\begin{aligned} B_{55} &= \frac{a_{44}}{\omega}, \quad B_{45} = -\frac{a_{45}}{\omega}, \quad B_{44} = \frac{a_{55}}{\omega}, \\ \omega &= a_{44}a_{55} - a_{45}^2, \end{aligned}\right\} \tag{2.10}$$

$$\left.\begin{aligned} A_1 &= B_{11}a_{13} + B_{12}a_{23} + B_{16}a_{36}, \\ A_2 &= B_{12}a_{13} + B_{22}a_{23} + B_{26}a_{36}, \\ A_3 &= B_{16}a_{13} + B_{26}a_{23} + B_{66}a_{36}. \end{aligned}\right\} \tag{2.11}$$

Using (2.1) and the equalities (1.2.2)

$$e_z = \frac{\partial u_z}{\partial z} = 0, \qquad u_z = u_z(x, y) = w(x, y), \tag{2.12}$$

i.e., just as in classical theory the displacements u_z for all points along a given normal plate element (the elements normal to the plate middle plane) are equal to the normal displacement w of the corresponding point of the middle plane.

From equations (1.4.5) and (2.2) we have the following formulas for the transverse shear deformations e_{xz} and e_{yz}

$$e_{yz} = a_{44}f_2(z)\psi + a_{45}f_1(z)\varphi + a_{44}Y_1 + a_{45}X_1 + \frac{z}{h}(a_{44}Y_2 + a_{45}X_2), \tag{2.13}$$

$$e_{xz} = a_{55}f_1(z)\varphi + a_{45}f_2(z)\psi + a_{55}X_1 + a_{45}Y_1 + \frac{z}{h}(a_{55}X_2 + a_{45}Y_2), \tag{2.14}$$

where the following notation is used:

$$X_1 = \frac{X^+ - X^-}{2}, \qquad Y_1 = \frac{Y^+ - Y^-}{2}, \tag{2.15}$$

$$X_2 = X^+ + X^-, \qquad Y_2 = Y^+ + Y^-. \tag{2.16}$$

From equations (1.2.2) with consideration for (2.12)

$$\frac{\partial u_x}{\partial z} = -\frac{\partial w}{\partial x} + e_{xz}, \qquad \frac{\partial u_y}{\partial z} = -\frac{\partial w}{\partial y} + e_{yz}. \tag{2.17}$$

Integration of the expression (2.17) with respect to z from zero to z with the condition that for z = 0, $u_x = u(x, y)$ and $u_y = v(x, y)$, the following values for the inplane displacements of any point in the plate are obtained:

$$u_x = u - z\frac{\partial w}{\partial x} + a_{55}J_{01}\varphi + a_{45}J_{02}\psi + z(a_{55}X_1 + a_{45}Y_1) + \\ + \frac{z^2}{2h}(a_{55}X_2 + a_{45}Y_2), \qquad (2.18)$$

$$u_y = v - z\frac{\partial w}{\partial y} + a_{44}J_{02}\psi + a_{45}J_{01}\varphi + z(a_{44}Y_1 + a_{45}X_1) + \\ + \frac{z^2}{2h}(a_{44}Y_2 + a_{45}X_2), \qquad (2.19)$$

where

$$J_{01} = \int_0^z f_1(z)\,dz, \qquad J_{02} = \int_0^z f_2(z)\,dz, \qquad (2.20)$$

$u(x, y)$, and $v(x, y)$ are inplane displacements of the corresponding point of the middle plane.

Equations (2.18) and (2.19) show that the inplane displacements u_x and u_y of any point of the plate located at a distance z from the middle plane along the normal have, in the general case, a non-linear dependence on z, which is contrary to the classical theory of plates.

Substituting values of u_x and u_y from (2.18) and (2.19) into (1.2.2) we obtain expressions for the corresponding deformation components:

$$e_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + a_{55}J_{01}\frac{\partial \varphi}{\partial x} + a_{45}J_{02}\frac{\partial \psi}{\partial x} + \\ + z\left(a_{55}\frac{\partial X_1}{\partial x} + a_{45}\frac{\partial Y_1}{\partial x}\right) + \frac{z^2}{2h}\left(a_{55}\frac{\partial X_2}{\partial x} + a_{45}\frac{\partial Y_2}{\partial x}\right), \qquad (2.21)$$

$$e_y = \frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + a_{44}J_{02}\frac{\partial \psi}{\partial y} + a_{45}J_{01}\frac{\partial \varphi}{\partial y} + \\ + z\left(a_{44}\frac{\partial Y_1}{\partial y} + a_{45}\frac{\partial X_1}{\partial y}\right) + \frac{z^2}{2h}\left(a_{44}\frac{\partial Y_2}{\partial y} + a_{45}\frac{\partial X_2}{\partial y}\right), \qquad (2.22)$$

$$e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{\partial x\,\partial y} + J_{01}\left(a_{55}\frac{\partial \varphi}{\partial y} + a_{45}\frac{\partial \varphi}{\partial x}\right) + \\ + J_{02}\left(a_{44}\frac{\partial \psi}{\partial x} + a_{45}\frac{\partial \psi}{\partial y}\right) + z\left[a_{55}\frac{\partial X_1}{\partial y} + a_{44}\frac{\partial Y_1}{\partial x} + a_{45}\left(\frac{\partial Y_1}{\partial y} + \frac{\partial X_1}{\partial x}\right)\right] + \\ + \frac{z^2}{2h}\left[a_{55}\frac{\partial X_2}{\partial y} + a_{44}\frac{\partial Y_2}{\partial x} + a_{45}\left(\frac{\partial Y_2}{\partial y} + \frac{\partial X_2}{\partial y}\right)\right]. \qquad (2.23)$$

Substituting values of τ_{xz} and τ_{yz} from (2.2) into the third equilibrium equation (1.2.3) in the absence of body forces ($P_z = 0$), as well as considering the equalities (2.15) and (2.16), and integrating with respect to z, the expression for normal stress σ_z is obtained:

$$\sigma_z = \chi(x, y) - J_{01}\frac{\partial \varphi}{\partial x} - J_{02}\frac{\partial \psi}{\partial y} - z\left(\frac{\partial X_1}{\partial x} + \frac{\partial Y_1}{\partial y}\right) - \\ - \frac{z^2}{2h}\left(\frac{\partial X_2}{\partial x} + \frac{\partial Y_2}{\partial y}\right), \qquad (2.24)$$

where $\chi(x, y)$ is the integration constant (with respect to z) and is determined from conditions on the plate surfaces which are of the following type:

$$\left.\begin{array}{lll} \text{at} & z = \dfrac{h}{2} & \sigma_z = Z^+; \\ \text{at} & z = -\dfrac{h}{2} & \sigma_z = -Z^-. \end{array}\right\} \qquad (2.25)$$

The following expression for $\chi(x, y)$ satisfies these conditions:

$$\chi(x, y) = K_1^+(h/2)\frac{\partial \varphi}{\partial x} + K_2^+(h/2)\frac{\partial \psi}{\partial y} + \\ + \frac{h}{8}\left(\frac{\partial X_2}{\partial x} + \frac{\partial Y_2}{\partial y}\right) + Z_1, \qquad (2.26)$$

as well as the following equation:

$$K_1^-(h/2)\frac{\partial \varphi}{\partial x} + K_2^-(h/2)\frac{\partial \psi}{\partial y} = -h\left(\frac{\partial X_1}{\partial x} + \frac{\partial Y_1}{\partial y}\right) - Z_2; \qquad (2.27)$$

where the following notation is used:

$$\left.\begin{array}{l} K_i^+\left(\dfrac{h}{2}\right) = \dfrac{1}{2}\left[J_{0i}\left(\dfrac{h}{2}\right) + J_{0i}\left(-\dfrac{h}{2}\right)\right], \\ K_i^-\left(\dfrac{h}{2}\right) = J_{0i}\left(\dfrac{h}{2}\right) - J_{0i}\left(-\dfrac{h}{2}\right), \end{array}\right\} \qquad (2.28)$$

$$Z_1 = \frac{Z^+ - Z^-}{2}, \quad Z_2 = Z^+ + Z^-. \qquad (2.29)$$

Equation (2.27), which is presented in terms of the functions φ and ψ, is obtained as a result of satisfaction of conditions on the surfaces (2.25). It is also considered as the third equation of equilibrium[6]. This will be self-evident from equations (3.3), (2.43) and (2.47).

Substituting values of e_x, e_y, e_{xy} and σ_z into formulas (2.3) – (2.5), equations are obtained for the principal stresses σ_x, σ_y, τ_{xy}:

$$\sigma_x = B_{11}\frac{\partial u}{\partial x} + B_{12}\frac{\partial v}{\partial y} + B_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) -$$
$$- z\left(B_{11}\frac{\partial^2 w}{\partial x^2} + B_{12}\frac{\partial^2 w}{\partial y^2} + 2B_{16}\frac{\partial^2 w}{\partial x\,\partial y}\right) +$$
$$+ J_{01}\left[(a_{55}B_{11} + a_{45}B_{16} + A_1)\frac{\partial \varphi}{\partial x} + (a_{45}B_{12} + a_{55}B_{16})\frac{\partial \varphi}{\partial y}\right] +$$
$$+ J_{02}\left[(a_{45}B_{11} + a_{44}B_{16})\frac{\partial \psi}{\partial x} + (a_{44}B_{12} + a_{45}B_{16} + A_1)\frac{\partial \psi}{\partial y}\right] -$$
$$- A_1\chi + z\left[B_{11}Q_1 + B_{12}Q_2 + B_{16}Q_3 + A_1\left(\frac{\partial X_1}{\partial x} + \frac{\partial Y_1}{\partial y}\right)\right] +$$
$$+ \frac{z^2}{2h}\left[B_{11}R_1 + B_{12}R_2 + B_{16}R_3 + A_1\left(\frac{\partial X_2}{\partial x} + \frac{\partial Y_2}{\partial y}\right)\right], \qquad (2.30)$$

$$\sigma_y = B_{22}\frac{\partial v}{\partial y} + B_{12}\frac{\partial u}{\partial x} + B_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) -$$
$$- z\left(B_{22}\frac{\partial^2 w}{\partial y^2} + B_{12}\frac{\partial^2 w}{\partial x^2} + 2B_{26}\frac{\partial^2 w}{\partial x\,\partial y}\right) +$$
$$+ J_{02}\left[(a_{44}B_{22} + a_{45}B_{26} + A_2)\frac{\partial \psi}{\partial y} + (a_{45}B_{12} + a_{44}B_{26})\frac{\partial \psi}{\partial x}\right] +$$
$$+ J_{01}\left[(a_{45}B_{22} + a_{55}B_{26})\frac{\partial \varphi}{\partial y} + (a_{55}B_{12} + a_{45}B_{26} + A_2)\frac{\partial \varphi}{\partial x}\right] -$$
$$- A_2\chi + z\left[B_{22}Q_2 + B_{12}Q_1 + B_{26}Q_3 + A_2\left(\frac{\partial X_1}{\partial x} + \frac{\partial Y_1}{\partial y}\right)\right] +$$
$$+ \frac{z^2}{2h}\left[B_{22}R_2 + B_{12}R_1 + B_{26}R_3 + A_2\left(\frac{\partial X_2}{\partial x} + \frac{\partial Y_2}{\partial y}\right)\right], \qquad (2.31)$$

$$\tau_{xy} = B_{16}\frac{\partial u}{\partial x} + B_{26}\frac{\partial v}{\partial y} + B_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) -$$
$$- z\left(B_{16}\frac{\partial^2 w}{\partial x^2} + B_{26}\frac{\partial^2 w}{\partial y^2} + 2B_{66}\frac{\partial^2 w}{\partial x\,\partial y}\right) +$$
$$+ J_{01}\left[(a_{55}B_{16} + a_{45}B_{66} + A_3)\frac{\partial \varphi}{\partial x} + (a_{45}B_{26} + a_{55}B_{66})\frac{\partial \varphi}{\partial y}\right] +$$
$$+ J_{02}\left[(a_{45}B_{16} + a_{44}B_{66})\frac{\partial \psi}{\partial x} + (a_{44}B_{26} + a_{45}B_{66} + A_3)\frac{\partial \psi}{\partial y}\right] -$$
$$- A_3\chi + z\left[B_{16}Q_1 + B_{26}Q_2 + B_{66}Q_3 + A_3\left(\frac{\partial X_1}{\partial x} + \frac{\partial Y_1}{\partial y}\right)\right] +$$
$$+ \frac{z^2}{2h}\left[B_{16}R_1 + B_{26}R_2 + B_{66}R_3 + A_3\left(\frac{\partial X_2}{\partial x} + \frac{\partial Y_2}{\partial y}\right)\right]; \qquad (2.32)$$

where the following notation is used:

$$\left.\begin{aligned} Q_1 &= a_{55}\frac{\partial X_1}{\partial x} + a_{45}\frac{\partial Y_1}{\partial x}, \quad Q_2 = a_{44}\frac{\partial Y_1}{\partial y} + a_{45}\frac{\partial X_1}{\partial y}, \\ Q_3 &= a_{55}\frac{\partial X_1}{\partial y} + a_{44}\frac{\partial Y_1}{\partial x} + a_{45}\left(\frac{\partial Y_1}{\partial y} + \frac{\partial X_1}{\partial x}\right), \end{aligned}\right\} \tag{2.33}$$

$$\left.\begin{aligned} R_1 &= a_{55}\frac{\partial X_2}{\partial x} + a_{45}\frac{\partial Y_2}{\partial x}, \quad R_2 = a_{44}\frac{\partial Y_2}{\partial y} + a_{45}\frac{\partial X_2}{\partial y}, \\ R_3 &= a_{55}\frac{\partial X_2}{\partial y} + a_{44}\frac{\partial Y_2}{\partial x} + a_{45}\left(\frac{\partial Y_2}{\partial y} + \frac{\partial X_2}{\partial x}\right). \end{aligned}\right\} \tag{2.34}$$

Equations (2.2), (2.24), (2.30) – (2.32) are the laws for stress variation through the plate thickness. However, just as in the classical theory, it is convenient to replace the stresses with statically equivalent stress resultants and moments acting over small areas of the normal plate cross sections.

Using basic equilibrium relationships for internal normal and tangential (T_x, T_y, $S = S_{xy} = S_{yx}$) and transverse (N_x, N_y) forces, as well as for bending (M_x, M_y) and torque ($H = H_{xy} = H_{yx}$) moments for unit lengths of the middle plane we obtain (see Figures 5 and 6):*

$$\begin{aligned} T_x = \int_{-h/2}^{h/2} \sigma_x\, dz &= C_{11}\frac{\partial u}{\partial x} + C_{12}\frac{\partial v}{\partial y} + C_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \\ &+ J_1\left[(a_{55}B_{11} + a_{45}B_{16} + A_1)\frac{\partial \varphi}{\partial x} + (a_{45}B_{12} + a_{55}B_{16})\frac{\partial \varphi}{\partial y}\right] + \\ &+ J_2\left[(a_{45}B_{11} + a_{44}B_{16})\frac{\partial \psi}{\partial x} + (a_{44}B_{12} + a_{45}B_{16} + A_1)\frac{\partial \psi}{\partial y}\right] - \\ &- A_1 h\chi + \frac{h^2}{24}\left[B_{11}R_1 + B_{12}R_2 + B_{16}R_3 + A_1\left(\frac{\partial X_2}{\partial x} + \frac{\partial Y_2}{\partial y}\right)\right], \end{aligned} \tag{2.35}$$

*Editor's comment: Note the difference between the usual English notation where N_x, N_y, N_{xy} designate the normal and tangential inplane stress resultants (as opposed to T_x, T_y, S used herein) Q_x and Q_y designate the transverse stress resultants (as opposed to N_x, N_y used herein) and M_x, M_y, M_{xy} designate the bending and torque moment resultants.

$$T_y = \int_{-h/2}^{h/2} \sigma_y \, dz = C_{22}\frac{\partial v}{\partial y} + C_{12}\frac{\partial u}{\partial x} + C_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) +$$

$$+ J_2\left[(a_{44}B_{22} + a_{45}B_{26} + A_2)\frac{\partial \psi}{\partial y} + (a_{45}B_{12} + a_{44}B_{26})\frac{\partial \psi}{\partial x}\right] +$$

$$+ J_1\left[(a_{45}B_{22} + a_{55}B_{26})\frac{\partial \varphi}{\partial y} + (a_{55}B_{12} + a_{45}B_{26} + A_2)\frac{\partial \varphi}{\partial x}\right] -$$

$$- A_2 h\chi + \frac{h^2}{24}\left[B_{22}R_2 + B_{12}R_1 + B_{26}R_3 + A_2\left(\frac{\partial X_2}{\partial x} + \frac{\partial Y_2}{\partial y}\right)\right], \qquad (2.36)$$

$$S = \int_{-h/2}^{h/2} \tau_{xy} \, dz = C_{16}\frac{\partial u}{\partial x} + C_{26}\frac{\partial v}{\partial y} + C_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) +$$

$$+ J_1\left[(a_{55}B_{16} + a_{45}B_{66} + A_3)\frac{\partial \varphi}{\partial x} + (a_{45}B_{26} + a_{55}B_{66})\frac{\partial \varphi}{\partial y}\right] +$$

$$+ J_2\left[(a_{45}B_{16} + a_{44}B_{66})\frac{\partial \psi}{\partial x} + (a_{44}B_{26} + a_{45}B_{66} + A_3)\frac{\partial \psi}{\partial y}\right] -$$

$$- A_3 h\chi + \frac{h^2}{24}\left[B_{16}R_1 + B_{26}R_2 + B_{66}R_3 + A_3\left(\frac{\partial X_2}{\partial x} + \frac{\partial Y_2}{\partial y}\right)\right], \qquad (2.37)$$

$$M_x = \int_{-h/2}^{h/2} \sigma_x z \, dz = -\left(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2} + 2D_{16}\frac{\partial^2 w}{\partial x \, \partial y}\right) +$$

$$+ J_3\left[(a_{55}B_{11} + a_{45}B_{16} + A_1)\frac{\partial \varphi}{\partial x} + (a_{45}B_{12} + a_{55}B_{16})\frac{\partial \varphi}{\partial y}\right] +$$

$$+ J_4\left[(a_{45}B_{11} + a_{44}B_{16})\frac{\partial \psi}{\partial x} + (a_{44}B_{12} + a_{45}B_{16} + A_1)\frac{\partial \psi}{\partial y}\right] +$$

$$+ \frac{h^3}{12}\left[B_{11}Q_1 + B_{12}Q_2 + B_{16}Q_3 + A_1\left(\frac{\partial X_1}{\partial x} + \frac{\partial Y_1}{\partial y}\right)\right], \qquad (2.38)$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y z \, dz = -\left(D_{22}\frac{\partial^2 w}{\partial y^2} + D_{12}\frac{\partial^2 w}{\partial x^2} + 2D_{26}\frac{\partial^2 w}{\partial x \, \partial y}\right) +$$

$$+ J_4\left[(a_{44}B_{22} + a_{45}B_{26} + A_2)\frac{\partial \psi}{\partial y} + (a_{45}B_{12} + a_{44}B_{26})\frac{\partial \psi}{\partial x}\right] +$$

$$+ J_3\left[(a_{45}B_{22} + a_{55}B_{26})\frac{\partial \varphi}{\partial y} + (a_{55}B_{12} + a_{45}B_{26} + A_2)\frac{\partial \varphi}{\partial x}\right] +$$

$$+ \frac{h^3}{12}\left[B_{22}Q_2 + B_{12}Q_1 + B_{26}Q_3 + A_2\left(\frac{\partial X_1}{\partial x} + \frac{\partial Y_1}{\partial y}\right)\right], \qquad (2.39)$$

$$H = \int_{-h/2}^{h/2} \tau_{xy} z\, dz = -\left(D_{16}\frac{\partial^2 w}{\partial x^2} + D_{26}\frac{\partial^2 w}{\partial y^2} + 2D_{66}\frac{\partial^2 w}{\partial x\,\partial y}\right) +$$
$$+ J_3\left[(a_{55}B_{16} + a_{45}B_{66} + A_3)\frac{\partial \varphi}{\partial x} + (a_{45}B_{26} + a_{55}B_{66})\frac{\partial \varphi}{\partial y}\right] +$$
$$+ J_4\left[(a_{45}B_{16} + a_{44}B_{66})\frac{\partial \psi}{\partial x} + (a_{44}B_{26} + a_{45}B_{66} + A_3)\frac{\partial \psi}{\partial y}\right] +$$
$$+ \frac{h^3}{12}\left[B_{16}Q_1 + B_{26}Q_2 + B_{66}Q_3 + A_3\left(\frac{\partial X_1}{\partial x} + \frac{\partial Y_1}{\partial y}\right)\right], \qquad (2.40)$$

$$N_x = \int_{-h/2}^{h/2} \tau_{xz} dz = J_5\varphi + hX_1, \qquad (2.41)$$

$$N_y = \int_{-h/2}^{h/2} \tau_{yz} dz = J_6\psi + hY_1; \qquad (2.42)$$

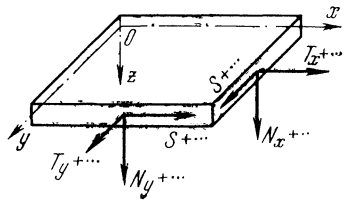

Figure 5.

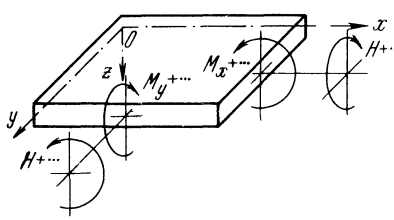

Figure 6.

Here the following notation is used:

$$\left.\begin{aligned} J_1 &= \int_{-h/2}^{h/2} J_{01}(z)\,dz, \quad J_2 = \int_{-h/2}^{h/2} J_{02}(z)\,dz, \\ J_3 &= \int_{-h/2}^{h/2} zJ_{01}(z)\,dz, \quad J_4 = \int_{-h/2}^{h/2} zJ_{02}(z)\,dz, \\ J_5 &= K_1^-\left(\frac{h}{2}\right) = \int_{-h/2}^{h/2} f_1(z)\,dz, \quad J_6 = K_2^-\left(\frac{h}{2}\right) = \int_{-h/2}^{h/2} f_2(z)\,dz, \end{aligned}\right\} \qquad (2.43)$$

as well as the known relationships for tension – compression (C_{ik}) and bending (D_{ik}) stiffnesses:

$$C_{ik} = hB_{ik}, \quad D_{ik} = \frac{h^3}{12} B_{ik}. \tag{2.44}$$

Integration of each equation of equilibrium in (1.2.3) excluding the presence of body forces, with respect to z within z = −h/2 to z = h/2 and multiplication of the first two equations of (1.2.3) by z within these same limits excluding again the body forces, and integrating this result with respect to z within the same limits, the following five plate differential equilibrium equations, as expressed with respect to the eight internal stress and moment resultants M_x, M_y, H, T_x, T_y, S, N_x, N_y, are obtained:

$$\frac{\partial T_x}{\partial x} + \frac{\partial S}{\partial y} = -X_2, \tag{2.45}$$

$$\frac{\partial T_y}{\partial y} + \frac{\partial S}{\partial x} = -Y_2, \tag{2.46}$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = -Z_2, \tag{2.47}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} = N_x - hX_1, \tag{2.48}$$

$$\frac{\partial M_y}{\partial y} + \frac{\partial H}{\partial x} = N_y - hY_1. \tag{2.49}$$

Surface boundary conditions have been used in the derivation of equations (2.45) – (2.49).

3. Equations of the Deformed Plate Surface and Boundary Conditions

It is evident from equations (2.35) – (2.42) that the internal stress and moment resultants of the plate are expressed in terms of displacements u, v, w and functions which arise in the expressions for the transverse shear, φ, ψ. These five functions can be shown to satisfy a system of differential equations of tenth order.

Substituting the values of the internal stress and moment resultants T_x, T_y, S, M_x, M_y, H, N_x, N_y into the equilibrium equations (2.45) – (2.49) the following tenth order system of differential equations of the deformed plate are obtained:

$$L_{11}(C_{ik})u + L_{12}(C_{ik})v + J_1 L_{14}\varphi + J_2 L_{12}\psi + \\ + (J_1 - hK_1^+)\frac{\partial}{\partial x} L_{14}\varphi + (J_2 - hK_2^+)\frac{\partial}{\partial y} L_{14}\psi = \\ = -\frac{h^2}{24}(L_1 X_2 + L_{12} Y_2) + \frac{h^2}{12}\left(\frac{\partial}{\partial x} L_{14} X_2 + \frac{\partial}{\partial y} L_{14} Y_2\right) + hL_{14} Z_1 - X_2, \tag{3.1}$$

$$L_{22}(C_{ik})v + L_{12}(C_{ik})u + J_2 L_2\psi + J_1 L_{21}\varphi + \\ + (J_2 - hK_2^+)\frac{\partial}{\partial y}L_{24}\psi + (J_1 - hK_1^+)\frac{\partial}{\partial x}L_{24}\varphi = \\ = -\frac{h^2}{24}(L_2 Y_2 + L_{21}X_2) + \frac{h^2}{12}\left(\frac{\partial}{\partial y}L_{24}Y_2 + \frac{\partial}{\partial y}L_{24}X_2\right) + hL_{24}Z_1 - Y_2, \tag{3.2}$$

$$J_5\frac{\partial\varphi}{\partial x} + J_6\frac{\partial\psi}{\partial y} = -Z_2 - h\left(\frac{\partial X_1}{\partial x} + \frac{\partial Y_1}{\partial y}\right), \tag{3.3}$$

$$L_{13}(D_{ik})w - J_3 L_1\varphi - J_4 L_{12}\psi + J_5\varphi - J_3\frac{\partial}{\partial x}L_{14}\varphi - J_4\frac{\partial}{\partial y}L_{14}\psi = \\ = \frac{h^3}{12}\left[L_1 X_1 + L_{12}Y_1 + \frac{\partial}{\partial x}L_{14}X_1 + \frac{\partial}{\partial y}L_{14}Y_1\right], \tag{3.4}$$

$$L_{23}(D_{ik})w - J_4 L_2\psi - J_3 L_{21}\varphi + J_6\psi - J_4\frac{\partial}{\partial y}L_{24}\psi - J_3\frac{\partial}{\partial x}L_{24}\varphi = \\ = \frac{h^3}{12}\left[L_2 Y_1 + L_{21}X_1 + \frac{\partial}{\partial y}L_{24}Y_1 + \frac{\partial}{\partial x}L_{24}X_1\right], \tag{3.5}$$

where the following notations are used for linear operators:

$$\left.\begin{aligned} L_{11}(C_{ik}) &= C_{11}\frac{\partial^2}{\partial x^2} + 2C_{16}\frac{\partial^2}{\partial x\,\partial y} + C_{66}\frac{\partial^2}{\partial y^2}, \\ L_{22}(C_{ik}) &= C_{22}\frac{\partial^2}{\partial y^2} + 2C_{26}\frac{\partial^2}{\partial x\,\partial y} + C_{66}\frac{\partial^2}{\partial x^2}, \\ L_{12}(C_{ik}) &= C_{16}\frac{\partial^2}{\partial x^2} + (C_{12} + C_{66})\frac{\partial^2}{\partial x\,\partial y} + C_{26}\frac{\partial^2}{\partial y^2}, \end{aligned}\right\} \tag{3.6}$$

$$\left.\begin{aligned} L_{13}(D_{ik}) &= D_{11}\frac{\partial^3}{\partial x^3} + 3D_{16}\frac{\partial^3}{\partial x^2\,\partial y} + \\ &\quad + (D_{12} + 2D_{66})\frac{\partial^3}{\partial x\,\partial y^2} + D_{26}\frac{\partial^3}{\partial y^3}, \\ L_{23}(D_{ik}) &= D_{22}\frac{\partial^3}{\partial y^3} + 3D_{26}\frac{\partial^3}{\partial y^2\,\partial x} + \\ &\quad + (D_{12} + 2D_{66})\frac{\partial^3}{\partial y\,\partial x^2} + D_{16}\frac{\partial^3}{\partial x^3}. \end{aligned}\right\} \tag{3.7}$$

$$\left.\begin{aligned}
L_1 &= (a_{55}B_{11} + a_{45}B_{16})\frac{\partial^2}{\partial x^2} + \\
&+ [a_{45}(B_{12} + B_{66}) + 2a_{55}B_{16}]\frac{\partial^2}{\partial x\,\partial y} + (a_{55}B_{66} + a_{45}B_{26})\frac{\partial^2}{\partial y^2}, \\
L_2 &= (a_{44}B_{22} + a_{45}B_{26})\frac{\partial^2}{\partial y^2} + \\
&+ [a_{45}(B_{12} + B_{66}) + 2a_{44}B_{26}]\frac{\partial^2}{\partial x\,\partial y} + (a_{44}B_{66} + a_{45}B_{16})\frac{\partial^2}{\partial x^2}, \\
L_{12} &= (a_{45}B_{11} + a_{44}B_{16})\frac{\partial^2}{\partial x^2} + \\
&+ [a_{44}(B_{12} + B_{66}) + 2a_{45}B_{16}]\frac{\partial^2}{\partial x\,\partial y} + (a_{45}B_{66} + a_{44}B_{26})\frac{\partial^2}{\partial y^2}, \\
L_{21} &= (a_{45}B_{22} + a_{55}B_{26})\frac{\partial^2}{\partial y^2} + \\
&+ [a_{55}(B_{12} + B_{66}) + 2a_{45}B_{26}]\frac{\partial^2}{\partial x\,\partial y} + (a_{45}B_{66} + a_{55}B_{16})\frac{\partial^2}{\partial x^2},
\end{aligned}\right\} \tag{3.8}$$

$$L_{14} = A_1\frac{\partial}{\partial x} + A_3\frac{\partial}{\partial y}, \quad L_{24} = A_2\frac{\partial}{\partial y} + A_3\frac{\partial}{\partial x}. \tag{3.9}$$

Equations (3.1) – (3.5) comprise the complete system of five differential equations with respect to the undetermined functions u, v, w, φ,ψ. Once these functions have been obtained and also using the equations from Section 2 of this chapter it is not difficult to compute basic stresses and internal stress and moment resultants.

The functions u, v, w, φ, ψ should satisfy both the system of differential equations (3.1) –(3.5) and the boundary conditions on the cylindrical surface C which forms the plate edge.

We assume for simplicity that the plate edge passes along the coordinate line x = 0, i.e., the surface of C represents the plane x = 0.

It is known that the uniform boundary conditions can be written in the following manner for the case of a three-dimensional problem in the theory of a thick plate [7, 8]:

(a) free edge

$$\sigma_x = 0, \quad \tau_{xy} = 0, \quad \tau_{xz} = 0; \tag{3.10}$$

(b) simply-supported edge

$$\left.\begin{aligned}
&\sigma_x = 0, \quad \tau_{xy} = 0, \quad u_z = 0, \\
&\sigma_x = 0, \quad u_y = 0, \quad u_z = 0;
\end{aligned}\right\} \tag{3.11}$$

(c) rigidly-fixed edge

$$\left.\begin{array}{lll} u_x = 0, & u_y = 0, & u_z = 0, \\ u_x = 0, & \tau_{xy} = 0, & u_z = 0. \end{array}\right\} \quad \text{or} \qquad (3.12)$$

The boundary conditions (3.10) – (3.12) cannot be fulfilled at each point of the boundary surface C in the case of arbitrarily selected functions $f_i(z)$. Due to this fact only "relaxed" boundary conditions are considered.

The principle of "relaxed" boundary conditions is justified by the theory suggested here, by St. Venant's principle, and by the methods used for enforcing actual boundary conditions.

The theory suggested in this book as well as all other improved theories of the same class[9, 10, 11] define more accurately the state of stress in the plate only at some distance from the boundary. In other words, they do not produce reliable information about the boundary conditions and the plate displacements from the point of view of the three-dimensional problem in the theory of elasticity. Nevertheless, it should be emphasized that they do define the principal state of stress of the plate more precisely than the classical theory[10, 11, 12, 13]

On the basis of the above, the "relaxed" boundary conditions of the problem (for the edge x = 0) to be used in place of (3.10) – (3.12) are:

(a) free edge. Considering equations (2.35) – (2.42) and (3.10)

$$\left.\begin{array}{ll} \int\limits_{-h/2}^{h/2} \sigma_x dz = T_x = 0, & \int\limits_{-h/2}^{h/2} \sigma_x z\, dz = M_x = 0, \\ \int\limits_{-h/2}^{h/2} \tau_{xz} dz = N_x = 0, & \int\limits_{-h/2}^{h/2} \tau_{xy} dz = S = 0, \\ & \int\limits_{-h/2}^{h/2} \tau_{xy} z\, dz = H = 0. \end{array}\right\} \qquad (3.13)$$

(b) Simply-supported edge. Using equations (2.12), (2.35) – (2.40) and (3.11) we have either:

$$\left.\begin{array}{lll} \int\limits_{-h/2}^{h/2} \sigma_x dz = T_x = 0, & \int\limits_{-h/2}^{h/2} \sigma_x z\, dz = M_x = 0, & u_z = w = 0, \\ \int\limits_{-h/2}^{h/2} \tau_{xy} dz = S = 0, & \int\limits_{-h/2}^{h/2} \tau_{xy} z\, dz = H = 0, & \end{array}\right\} \qquad (3.14)$$

or by assuming [14] in (3.11) that the condition $u_y = 0$ is satisfied only for $z = \pm z_0$ ($0 \leqslant z_0 \leqslant h/2$) we have the following expression from the second line of (3.11):

$$\left.\begin{aligned} &\int_{-h/2}^{h/2} \sigma_x dz = T_x = 0, \quad \int_{-h/2}^{h/2} \sigma_x z\, dz = M_x = 0, \quad u_z = w = 0, \\ &v + a_{44}K_2^+(z_0)\psi + a_{45}K_1^+(z_0)\varphi + \frac{z_0^2}{2h}(a_{44}Y_2 + a_{45}X_2) = 0, \\ &-\frac{\partial w}{\partial y} + a_{44}\frac{K_2^-(z_0)}{2z_0}\psi + a_{45}\frac{K_1(z_0)}{2z_0}\varphi + \\ &\qquad + (a_{44}Y_1 + a_{45}X_1) = 0. \end{aligned}\right\} \quad (3.14')$$

(c) Rigidly-fixed edge. Just as in the case of the simply-supported edge, we assume that conditions (3.12) are fulfilled only for $z = \pm z_0$ ($0 \leqslant z_0 \leqslant h/2$) and because of (2.12), (2.18), (2.19) and (3.12):

$$u_x|_{z=\pm z_0} = 0, \quad u_y|_{z=\pm z_0} = 0, \quad w = 0,$$

or

$$\left.\begin{aligned} &u + \frac{z_0^2}{2h}(a_{55}X_2 + a_{45}Y_2) + a_{55}K_1^+(z_0)\varphi + \\ &\qquad + a_{45}K_2^+(z_0)\psi = 0, \quad u_z = w = 0, \\ &v + \frac{z_0^2}{2h}(a_{44}Y_2 + a_{45}X_2) + a_{44}K_2^+(z_0)\psi + a_{45}K_1^+(z_0)\varphi = 0, \\ &-\frac{\partial w}{\partial x} + a_{55}\frac{K_1^-(z_0)}{2z_0}\varphi + a_{45}\frac{K_2^-(z_0)}{2z_0}\psi + (a_{55}X_1 + a_{45}Y_1) = 0, \\ &-\frac{\partial w}{\partial y} + a_{44}\frac{K_2^-(z_0)}{2z_0}\psi + a_{45}\frac{K_1^-(z_0)}{2z_0}\varphi + (a_{44}Y_1 + a_{45}X_1) = 0. \end{aligned}\right\} \quad (3.15)$$

The second conditions for a rigidly fixed edge on the basis of the second line in (3.12), proceeding as in preceding cases, are:

$$\left.\begin{aligned} &u + a_{55}K_1^+(z_0)\varphi + a_{45}K_2(z_0)\psi + \frac{z_0^2}{2h}(a_{55}X_2 + a_{45}Y_2) = 0, \\ &\qquad u_z = w = 0, \\ &-\frac{\partial w}{\partial x} + a_{55}\frac{K_1^-(z_0)}{2z_0}\varphi + a_{45}\frac{K_2^-(z_0)}{2z_0}\psi + (a_{55}X_1 + a_{45}Y_1) = 0, \\ &\int_{-h/2}^{h/2} \tau_{xy} dz = S = 0, \quad \int_{-h/2}^{h/2} \tau_{xy} z\, dz = H = 0. \end{aligned}\right\} \quad (3.15')$$

Mixed boundary conditions can also be presented as well as the uniform boundary conditions (3.13) – (3.15):

(a) Loaded edge:

$$T_x = T_x^*, \quad M_x = M_x^*, \quad N_x = N_x^*, \quad S = S^*, \quad H = H^*, \quad (3.16)$$

where $T^*_x \ldots$, H^* are forces (stress resultants) applied to the edge under consideration; in particular cases some of these stress resultants could equal zero.

(b) Simply-supported edge under the action of stress and moment resultants:

$$T_x = T_x^*, \quad M_x = M_x^*, \quad w = 0, \quad S = S^*, \quad H = H^*; \quad (3.17)$$

(c) Displaced edge:

$$\left.\begin{aligned}
&u + a_{55}K_1^+(z_0)\varphi + a_{45}K_2^+(z_0)\psi + \\
&\qquad + \frac{z_0^2}{2h}(a_{55}X_2 + a_{45}Y_2) = \frac{u^+ + u^-}{2}, \quad w = w^*, \\
&v + a_{44}K_2^+(z_0)\psi + a_{45}K_1^+(z_0)\varphi + \\
&\qquad + \frac{z_0^2}{2h}(a_{44}Y_2 + a_{45}X_2) = \frac{v^+ + v^-}{2}, \\
&-\frac{\partial w}{\partial x} + a_{55}\frac{K_1^-(z_0)}{2z_0}\varphi + a_{45}\frac{K_2^-(z_0)}{2z_0}\psi + \\
&\qquad + (a_{55}X_1 + a_{45}Y_1) = \frac{u^+ - u^-}{2z_0}, \\
&-\frac{\partial w}{\partial y} + a_{44}\frac{K_2^-(z_0)}{2z_0}\psi + a_{45}\frac{K_1^-(z_0)}{2z_0}\varphi + \\
&\qquad + (a_{44}Y_1 + a_{45}X_1) = \frac{v^+ - v^-}{2z_0};
\end{aligned}\right\} \quad (3.18)$$

where w^* is the given normal edge displacement, u^+, u^-, v^+, v^- are given inplane displacements of the edge which correspond to $z = \pm z_0$. In particular some of these specified displacements could equal zero.

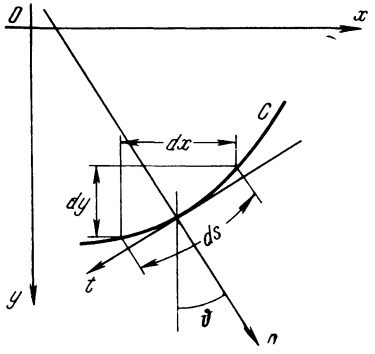

Figure 7.

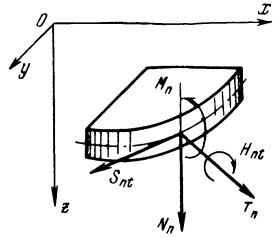

Figure 8.

Many types of plate supports are encountered in actual structures and a multiplicity of boundary supports for actual plates cannot be represented with sufficient accuracy with the present mathematical models and boundary conditions. Due to this fact only a few possible combinations of boundary conditions have been given here.

In the general case it is assumed that the plate edge is curvilinear and has an external normal n (Figure 7). Then the internal stress and moment resultants with respect to the curvilinear edge (Figure 8) can be written as follows:

$$\begin{aligned}
T_n &= T_x \sin^2 \vartheta + T_y \cos^2 \vartheta + 2S \sin \vartheta \cos \vartheta, \\
S_{nt} &= S(\sin^2 \vartheta - \cos^2 \vartheta) + (T_y - T_x) \sin \vartheta \cos \vartheta, \\
N_n &= N_x \cos \vartheta + N_y \sin \vartheta, \\
M_n &= M_x \sin^2 \vartheta + M_y \cos^2 \vartheta + 2H \sin \vartheta \cos \vartheta, \\
H_{nt} &= H(\sin^2 \vartheta - \cos^2 \vartheta) + (M_y - M_x) \sin \vartheta \cos \vartheta,
\end{aligned}$$

For normal and tangential displacement components of any point of the plate edge within a plane parallel to the middle plane, we have:

$$u_n = u_x \sin \vartheta + u_y \cos \vartheta, \qquad u_{nt} = u_y \sin \vartheta - u_x \cos \vartheta.$$

On the basis of the above relationships, the following uniform boundary conditions for the curvilinear edge of the plate can be stated:

(a) Free edge

$$T_n = 0, \quad M_n = 0, \quad N_n = 0, \quad S_{nt} = 0, \quad H_{nt} = 0; \tag{3.19}$$

(b) Simply-supported edge

$$T_n = 0, \quad M_n = 0, \quad S_{nt} = 0, \quad H_{nt} = 0, \quad w = 0, \tag{3.20}$$

or

$$T_n = 0, \quad M_n = 0, \quad w = 0, \quad u_{nt}|_{z=\pm z_0} = 0; \tag{3.20'}$$

(c) Rigidly fixed edge

$$u_n|_{z=\pm z_0} = 0, \quad u_{nt}|_{z=\pm z_0} = 0, \quad w = 0, \tag{3.21}$$

or

$$u_n|_{z=\pm z_0} = 0, \quad S_{nt} = 0, \quad H_{nt} = 0, \quad w = 0. \tag{3.21'}$$

The mixed boundary conditions for the curvilinear edge may be presented similarly to conditions (3.16) – (3.18).

Assuming that $a_{44} = a_{55} = a_{45} = 0$, $A_1 = A_2 = A = 0$, $X^+ = X^- = Y^+ = Y^- = 0$ in the equations and formulas given in the second and third sections of this

chapter and performing certain necessary transformations, the equations and identities of the classical theory of anisotropic plates are obtained[15]. These simplifications will be used below and each time the corresponding equations of the classical theory of plates will be obtained.

4. Selection of the Functions $f_i(z)$

One of the principal steps in the improved theory of anisotropic plates presented in the previous sections involves the selection of the functions $f_i(z)$.

It is evident from the results of certain researchers[16,17] that some inaccuracies which are allowed during the selection of the functions $f_i(z)$ do not have a major influence on the final results as computed at a distance from the boundaries. Arbitrariness in the logical selection of the functions $f_i(z)$ cannot introduce inadmissible errors into the suggested theory. This is amply illustrated on appropriate examples in Section 2 of Chapter III and in Section 6 of Chapter IV.

However, the selection of the functions $f_i(z)$ should preferably be based on the analysis of the shear stress distribution τ_{xz} and τ_{yz} using the sufficiently accurate theories of the bending of thick plates[18–22]. This analysis shows that tangential stresses τ_{xz} and τ_{yz} for the case of thick as well as thin plates have a parabolic distribution.

It is of interest to note here that in the solution of the plane bending problem of a strip fixed at one end[23] the shear stress τ_{xz} (which varies at the fixed end according to a complex law) is smoothed out at a distance of h/5 from the fixed end and at a distance of h/2 the distribution is parabolic.

Following the above discussion and considering that we are interested in this book in transverse bending of plates and in phenomena related to transverse bending of plates, it is assumed that

$$f(z) = f_1(z) = f_2(z) = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right). \tag{4.1}$$

From (4.1), (2.20), (2.28) and (2.43) we obtain

$$\left.\begin{aligned} J_0 = J_{01} = J_{02} = \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right), \quad J_1 = J_2 = 0, \\ J_3 = J_4 = \frac{h^5}{120}, \quad J_5 = J_6 = K_i^-\left(\frac{h}{2}\right) = \frac{h^3}{12}, \\ K_i^-(z_0) = z_0\left(\frac{h^2}{4} - \frac{z_0^2}{3}\right), \quad K_i^+ = 0. \end{aligned}\right\} \tag{4.2}$$

However, while considering (4.1) for f(z), we do not deny the possibility of selecting other logical distributions for the functions $f_i(z)$.

5. Equation of the Deformed Plate Surface and Boundary Conditions
(Continuation of Section 3)

For the case when the plate is made of an orthotropic material and the direction of the x, y and z axes coincide with the principal directions of elasticity the governing equations of the problem are simplified.

Considering equations (1.4.7) – (1.4.9), the elastic constants of an orthotropic material are:

$$\left.\begin{aligned}
&a_{11} = \frac{1}{E_1}, \quad a_{22} = \frac{1}{E_2}, \quad a_{33} = \frac{1}{E_3},\\
&a_{12} = -\frac{\nu_{12}}{E_2} = -\frac{\nu_{21}}{E_1} = -\frac{\nu_2}{E_2} = -\frac{\nu_1}{E_1},\\
&a_{13} = -\frac{\nu_{13}}{E_3} = -\frac{\nu_{31}}{E_1}, \quad a_{23} = -\frac{\nu_{23}}{E_3} = -\frac{\nu_{32}}{E_2},\\
&a_{44} = \frac{1}{G_{23}}, \quad a_{55} = \frac{1}{G_{13}}, \quad a_{66} = \frac{1}{G_{12}},\\
&a_{16} = 0, \quad a_{26} = 0, \quad a_{36} = 0, \quad a_{45} = 0.
\end{aligned}\right\} \qquad (5.1)$$

Substituting these values of a_{ik} into (2.8) – (2.11), we obtain the coefficients B_{ik} and A_i:

$$\left.\begin{aligned}
&B_{11} = \frac{a_{22}}{\Omega_0} = \frac{E_1}{1-\nu_1\nu_2}, \quad B_{22} = \frac{a_{11}}{\Omega_0} = \frac{E_2}{1-\nu_1\nu_2},\\
&B_{12} = -\frac{a_{12}}{\Omega_0} = \frac{\nu_2 E_1}{1-\nu_1\nu_2} = \frac{\nu_1 E_2}{1-\nu_1\nu_2}, \quad \nu_1 = \nu_{21}, \ \nu_2 = \nu_{12},\\
&B_{66} = \frac{1}{a_{66}} = G_{12}, \quad B_{55} = \frac{1}{a_{55}} = G_{13}, \quad B_{44} = \frac{1}{a_{44}} = G_{23},\\
&A_1 = B_{11}a_{13} + B_{12}a_{23} = -\frac{E_1}{E_3}\frac{\nu_{13}+\nu_2\nu_{23}}{1-\nu_1\nu_2} =\\
&\qquad = -\frac{\nu_{31}}{\nu_{13}}\frac{\nu_{13}+\nu_2\nu_{23}}{1-\nu_1\nu_2},\\
&A_2 = B_{12}a_{13} + B_{22}a_{23} = \frac{E_2}{E_3}\frac{\nu_{23}+\nu_1\nu_{13}}{1-\nu_1\nu_2} = -\frac{\nu_{32}}{\nu_{23}}\frac{\nu_{23}+\nu_1\nu_{13}}{1-\nu_1\nu_2},\\
&A_3 = 0, \qquad \Omega_0 = a_{11}a_{22} - a_{12}^2.
\end{aligned}\right\} \qquad (5.2)$$

On the assumption that $X^{\pm} = 0$, $Y^{\pm} = 0$ (the case when the plate is loaded only by a normally applied force $Z^{\pm}$ is of great practical interest) and considering (4.1), (4.2) the following system of differential equations is obtained from equations (3.1) – (3.5) for the orthotropic plate:

$$C_{11}\frac{\partial^2 u}{\partial x^2} + C_{66}\frac{\partial^2 u}{\partial y^2} + (C_{12}+C_{66})\frac{\partial^2 v}{\partial x\,\partial y} = A_1 h\frac{\partial Z_1}{\partial x}, \qquad (5.3)$$

$$C_{22}\frac{\partial^2 v}{\partial y^2}+C_{66}\frac{\partial^2 v}{\partial x^2}+(C_{12}+C_{66})\frac{\partial^2 u}{\partial x\,\partial y}=A_2 h\frac{\partial Z_1}{\partial y}, \tag{5.4}$$

$$\frac{\partial \varphi}{\partial x}+\frac{\partial \psi}{\partial y}=-\frac{12}{h^3}Z_2, \tag{5.5}$$

$$D_{11}\frac{\partial^3 w}{\partial x^3}+(D_{12}+2D_{66})\frac{\partial^3 w}{\partial x\,\partial y^2}-\frac{h^2}{10}\left[a_{55}\left(D_{11}\frac{\partial^2}{\partial x^2}+D_{66}\frac{\partial^2}{\partial y^2}\right)\varphi+\right.$$

$$\left.+a_{44}(D_{12}+D_{66})\frac{\partial^2 \psi}{\partial x\,\partial y}\right]+\frac{h^3}{12}\varphi=-A_1\frac{h^2}{10}\frac{\partial Z_2}{\partial x}, \tag{5.6}$$

$$D_{22}\frac{\partial^3 w}{\partial y^3}+(D_{12}+2D_{66})\frac{\partial^3 w}{\partial y\,\partial x^2}-\frac{h^2}{10}\left[a_{44}\left(D_{22}\frac{\partial^2}{\partial y^2}+D_{66}\frac{\partial^2}{\partial x^2}\right)\psi+\right.$$

$$\left.+a_{55}(D_{12}+D_{66})\frac{\partial^2 \varphi}{\partial x\,\partial y}\right]+\frac{h^3}{12}\psi=-A_2\frac{h^2}{10}\frac{\partial Z_2}{\partial y} \tag{5.7}$$

For the internal stress and moment resultants, and for the stresses, we obtain equations corresponding to the system (5.3) – (5.7) from equations (2.35) – (2.41) and (2.30) – (2.32).

$$T_x=C_{11}\frac{\partial u}{\partial x}+C_{12}\frac{\partial v}{\partial y}-hA_1Z_1, \tag{5.8}$$

$$T_y=C_{22}\frac{\partial v}{\partial y}+C_{12}\frac{\partial u}{\partial x}-hA_2Z_1, \tag{5.9}$$

$$S=C_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \tag{5.10}$$

$$M_x=-D_{11}\frac{\partial^2 w}{\partial x^2}-D_{12}\frac{\partial^2 w}{\partial y^2}+\frac{h^2}{10}\left(a_{55}D_{11}\frac{\partial \varphi}{\partial x}+a_{44}D_{12}\frac{\partial \psi}{\partial y}\right)-\frac{h^2}{10}A_1Z_2, \tag{5.11}$$

$$M_y=-D_{22}\frac{\partial^2 w}{\partial y^2}-D_{12}\frac{\partial^2 w}{\partial x^2}+\frac{h^2}{10}\left(a_{44}D_{22}\frac{\partial \psi}{\partial y}+a_{55}D_{12}\frac{\partial \varphi}{\partial x}\right)-\frac{h^2}{10}A_2Z_2, \tag{5.12}$$

$$H=-2D_{66}\frac{\partial^2 w}{\partial x\,\partial y}+\frac{h^2}{10}D_{66}\left(a_{55}\frac{\partial \varphi}{\partial y}+a_{44}\frac{\partial \psi}{\partial x}\right), \tag{5.13}$$

$$N_x=\frac{h^3}{12}\varphi,\quad N_y=\frac{h^3}{12}\psi, \tag{5.14}$$

$$\sigma_x=B_{11}\left(\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}\right)+B_{12}\left(\frac{\partial v}{\partial y}-z\frac{\partial^2 w}{\partial y^2}\right)+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\times$$

$$\times\left(a_{55}B_{11}\frac{\partial \varphi}{\partial x}+a_{44}B_{12}\frac{\partial \psi}{\partial y}\right)-A_1\left[Z_1+\left(\frac{3}{2}\frac{z}{h}-2\frac{z^3}{h^3}\right)Z_2\right], \tag{5.15}$$

$$\sigma_y = B_{22}\left(\frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2}\right) + B_{12}\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right) + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\times$$
$$\times\left(a_{44}B_{22}\frac{\partial \psi}{\partial y} + a_{55}B_{12}\frac{\partial \varphi}{\partial x}\right) - A_2\left[Z_1 + \left(\frac{3}{2}\frac{z}{h} - 2\frac{z^3}{h^3}\right)Z_2\right], \quad (5.16)$$

$$\tau_{xy} = B_{66}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{\partial x\,\partial y} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(a_{55}\frac{\partial \varphi}{\partial y} + a_{44}\frac{\partial \psi}{\partial x}\right)\right]. \quad (5.17)$$

Equation (5.5) and the values of the elastic constants given at the beginning of this section were used in the derivation of equations (5.3) – (5.7) and the identities (5.8) – (5.17).

A close examination of (3.1) – (3.5) and (5.3) – (5.7) reveals that the complete system of differential equations (3.1) – (3.5) can be divided into two independent systems. These systems are equations (5.3), (5.4), which represent some generalized plane problem (which is of no interest to us) and the equations (5.5) – (5.7) which represent the problem of transverse bending of the plate.

The general theory of anisotropic plates presented in preceding sections of this chapter and at the beginning of this section can be somewhat simplified. Only the transverse bending problem of orthotropic plates will be considered.

As opposed to the basic theory, one additional assumption is used in addition to the initial assumptions (a) and (b) (see Section 1). Namely, the normal stress σ_z on areas parallel to the middle plane is considered negligible with respect to all other stresses.

Formally this assumption is equivalent to $A_1 = 0$, $A_2 = 0$. Using this result and the equalities (5.11) – (5.17) we obtain the following expressions for internal moment resultants, transverse stress resultants, and stresses:

$$M_x = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} + \frac{h^2}{10}\left(a_{55}D_{11}\frac{\partial \varphi}{\partial x} + a_{44}D_{12}\frac{\partial \psi}{\partial y}\right), \quad (5.18)$$

$$M_y = -D_{22}\frac{\partial^2 w}{\partial y^2} - D_{12}\frac{\partial^2 w}{\partial x^2} + \frac{h^2}{10}\left(a_{44}D_{22}\frac{\partial \psi}{\partial y} + a_{55}D_{12}\frac{\partial \varphi}{\partial x}\right), \quad (5.19)$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial x\,\partial y} + \frac{h^2}{10}D_{66}\left(a_{55}\frac{\partial \varphi}{\partial y} + a_{44}\frac{\partial \psi}{\partial x}\right), \quad (5.20)$$

$$N_x = \frac{h^3}{12}\varphi, \quad N_y = \frac{h^3}{12}\psi, \quad (5.21)$$

$$\sigma_x = -z\left(B_{11}\frac{\partial^2 w}{\partial x^2} + B_{12}\frac{\partial^2 w}{\partial y^2}\right) +$$
$$+ \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(a_{55}B_{11}\frac{\partial \varphi}{\partial x} + a_{44}B_{12}\frac{\partial \psi}{\partial y}\right), \quad (5.22)$$

$$\sigma_y = -z\left(B_{22}\frac{\partial^2 w}{\partial y^2} + B_{12}\frac{\partial^2 w}{\partial x^2}\right) + \\ + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(a_{44}B_{22}\frac{\partial \psi}{\partial y} + a_{55}B_{12}\frac{\partial \varphi}{\partial x}\right), \qquad (5.23)$$

$$\tau_{xy} = -2zB_{66}\frac{\partial^2 w}{\partial x\,\partial y} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)B_{66}\left(a_{55}\frac{\partial \varphi}{\partial y} + a_{44}\frac{\partial \psi}{\partial x}\right). \qquad (5.24)$$

Finally, from equations (5.5) – (5.7) a complete system of three differential equations with respect to the three undetermined functions w, φ, ψ is obtained:

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} = -\frac{12}{h^3}Z_2, \qquad (5.25)$$

$$D_{11}\frac{\partial^3 w}{\partial x^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x\,\partial y^2} - \frac{h^2}{10}\left[a_{55}D_{11}\frac{\partial^2 \varphi}{\partial x^2} + \right. \\ \left. + a_{55}D_{66}\frac{\partial^2 \varphi}{\partial y^2} + a_{44}(D_{12} + D_{66})\frac{\partial^2 \psi}{\partial x\,\partial y}\right] + \frac{h^3}{12}\varphi = 0, \qquad (5.26)$$

$$D_{22}\frac{\partial^3 w}{\partial y^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial y\,\partial x^2} - \frac{h^2}{10}\left[a_{44}D_{22}\frac{\partial^2 \psi}{\partial y^2} + \right. \\ \left. + a_{44}D_{66}\frac{\partial^2 \psi}{\partial x^2} + a_{55}(D_{12} + D_{66})\frac{\partial^2 \varphi}{\partial x\,\partial y}\right] + \frac{h^3}{12}\psi = 0. \qquad (5.27)$$

We assume that

$$w = \left\{\frac{h^4}{100}a_{44}a_{55}\left[D_{11}D_{66}\frac{\partial^4}{\partial x^4} + (D_{11}D_{22} - 2D_{12}D_{66} - D_{12}^2)\frac{\partial^4}{\partial x^2\,\partial y^2} + \right.\right. \\ \left. + D_{22}D_{66}\frac{\partial^4}{\partial y^4}\right] - \frac{h^5}{120}\left[(a_{55}D_{11} + a_{44}D_{66})\frac{\partial^2}{\partial x^2} + \right. \\ \left.\left. + (a_{44}D_{22} + a_{55}D_{66})\frac{\partial^2}{\partial y^2}\right] + \frac{h^6}{144}\right\}\Phi, \qquad (5.28)$$

$$\varphi = \left\{\frac{h^2}{10}a_{44}\left[D_{11}D_{66}\frac{\partial^5}{\partial x^5} + (D_{11}D_{22} - 2D_{12}D_{66} - D_{12}^2)\frac{\partial^5}{\partial x^3\,\partial y^2} + \right.\right. \\ \left. + D_{22}D_{66}\frac{\partial^5}{\partial x\,\partial y^4}\right] - \frac{\partial^3}{12}\left[D_{11}\frac{\partial^3}{\partial x^3} + (D_{12} + 2D_{66})\frac{\partial^3}{\partial x\,\partial y^2}\right]\Big\}\Phi, \qquad (5.29)$$

$$\psi = \left\{\frac{h^2}{10}a_{55}\left[D_{22}D_{66}\frac{\partial^5}{\partial y^5} + (D_{11}D_{22} - 2D_{12}D_{66} - D_{12}^2)\frac{\partial^5}{\partial y^3\,\partial x^2} + \right.\right. \\ \left. + D_{11}D_{66}\frac{\partial^5}{\partial y\,\partial x^4}\right] - \frac{h^3}{12}\left[D_{22}\frac{\partial^3}{\partial y^3} + (D_{12} + 2D_{66})\frac{\partial^3}{\partial y\,\partial x^2}\right]\Big\}\Phi, \qquad (5.30)$$

which identically satisfy equations (5.26) and (5.27). And from (5.25) a differential equation of sixth order with respect to the undetermined function $\Phi = \Phi(x, y)$ is then obtained:

$$\left[D_{11}\frac{\partial^4}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4}{\partial y^4}\right]\Phi -$$

$$- \frac{12}{h^3}\frac{h^2}{10}\left\{a_{44}D_{11}D_{66}\frac{\partial^6}{\partial x^6} + [a_{44}(D_{11}D_{22} - 2D_{12}D_{66} - D_{12}^2) + \right.$$

$$+ a_{55}D_{11}D_{66}]\frac{\partial^6}{\partial x^4 \partial y^2} + [a_{55}(D_{11}D_{22} - 2D_{12}D_{66} - D_{12}^2) +$$

$$\left. + a_{44}D_{22}D_{66}]\frac{\partial^6}{\partial y^4 \partial x^2} + a_{55}D_{22}D_{66}\frac{\partial^6}{\partial y^6}\right\}\Phi = \frac{144}{h^6}Z_2. \qquad (5.31)$$

Thus, the problem of transverse bending of an orthotropic plate is reduced to the determination of some potential function $\Phi(x, y)$ which satisfies the differential equation (5.31) and boundary conditions given in Section 3 of this Chapter.

Substituting values of w, φ, ψ from (5.28), (5.29) and (5.30), respectively, into (5.18) – (5.24), it is quite straightforward to represent internal stress and moment resultants and stresses through the undetermined function Φ. These formulas are not given here because of their length. In general, it is easy to find all desired values of these quantities from equations (5.18) – (5.24) and (5.28) – (5.30) when values of the function Φ are known.

A close examination of the derivation of equations (5.18) – (5.31) shows that analogous equations for the bending of a plate with a more general (as opposed to orthotropic) case of anisotropy can be derived from (2.30) – (2.44) and (3.1) – (3.8) if we assume that $A_1 = 0$, $A_2 = 0$; $A_3 = 0$.

It can be shown that in this case we are inconsistent in the exlusion of normal stresses σ_z. The fact is that a correction to the theory of nondeformable normals, which is introduced by the consideration of stresses σ_z, is approximately of the same order as the correction which is introduced by the consideration of transverse shears, and this latter correction is still taken into account when we assume that $A_1 = 0$, $A_2 = 0$. This inconsistency will not introduce inadmissible errors in the majority of actual problems, as has been demonstrated before [24, 25] and will be further shown in this book.

6. Particular Theory [27–29]

An additional theory is presented in this section which is more accurate than the classical theory of bending of anisotropic plates made of material which has only one plane of elastic symmetry which is parallel to the plate middle plane.

It is assumed that the plate is acted on by only loads applied normal to the

plate middle plane $Z^{\pm} \neq 0$ ($X^{\pm} = 0$, $Y^{\pm} = 0$).

This theory is based on the following assumptions:

(a) the displacement u_z which is normal to the plate middle plane does not depend on the z coordinate;

(b) it is considered that for the determination of deformations the shear stresses τ_{xz}, τ_{yz} and the normal stress σ_z do not differ from the corresponding stresses found with the use of the hypothesis of non-deformable normals, i.e., from the corresponding stresses of the classical theory of bending of anisotropic plates.

On the basis of hypotheses (a) and (b) we assume the following approximate relationships:

$$e_z = 0, \quad \sigma_z = Z_1 + \left(\frac{3}{2}\frac{z}{h} - 2\frac{z^3}{h^3}\right) Z_2, \tag{6.1}$$

$$\tau_{xz} = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right)\varphi_0, \quad \tau_{yz} = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right)\psi_0, \tag{6.2}$$

where, as is known [30]

$$\varphi_0 = -\left[B_{11}\frac{\partial^3 w_0}{\partial x^3} + 3B_{16}\frac{\partial^3 w_0}{\partial x^2\,\partial y} + (B_{12} + 2B_{66})\frac{\partial^3 w_0}{\partial x\,\partial y^2} + B_{26}\frac{\partial^3 w_0}{\partial y^3}\right], \tag{6.3}$$

$$\psi_0 = -\left[B_{22}\frac{\partial^3 w_0}{\partial y^3} + 3B_{26}\frac{\partial^3 w_0}{\partial y^2\,\partial x} + (B_{12} + 2B_{66})\frac{\partial^3 w_0}{\partial y\,\partial x^2} + B_{16}\frac{\partial^3 w_0}{\partial x^3}\right], \tag{6.4}$$

w_0 is the normal displacement of the corresponding anisotropic plate as found with the classical theory, i.e., from the differential equation

$$D_{11}\frac{\partial^4 w_0}{\partial x^4} + 4D_{16}\frac{\partial^4 w_0}{\partial x^3\,\partial y} + 2(D_{12} + 2D_{66})\frac{\partial^4 w_0}{\partial x^2\,\partial y^2} + 4D_{26}\frac{\partial^4 w_0}{\partial x\,\partial y^3} + D_{22}\frac{\partial^4 w_0}{\partial y^4} = Z_2$$

and satisfying the corresponding boundary conditions.

Equations (2.24) – (2.27) as reduced to the classical theory were used here to determine σ_z. Namely, the equations

$$\sigma_z^0 = \chi^0(x, y) - \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(\frac{\partial \varphi_0}{\partial x} + \frac{\partial \psi_0}{\partial y}\right),$$

$$\chi^0(x, y) = Z_1, \quad \frac{\partial \varphi_0}{\partial x} + \frac{\partial \psi_0}{\partial y} = -\frac{12}{h^3}Z_2,$$

as well as the equilibrium conditions on the surface (2.25) and the notation (2.29) were used to obtain the result (6.1).

Following the usual method (see Section 2) the following expressions are obtained for the displacement of any point of the plate subjected to bending:

$$u_z = w(x, y), \tag{6.5}$$

$$u_x = -z\frac{\partial w}{\partial x} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)(a_{55}\varphi_0 + a_{45}\psi_0), \tag{6.6}$$

$$u_y = -z\frac{\partial w}{\partial y} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)(a_{44}\psi_0 + a_{45}\varphi_0), \tag{6.7}$$

where w = w(x, y) is the undetermined normal displacement of the middle plane.

From (2.3) – (2.5) with consideration of (1.2.2), (6.1), (6.5) – (6.7) we obtain for the stresses:

$$\sigma_x = -z\left(B_{11}\frac{\partial^2 w}{\partial x^2} + B_{12}\frac{\partial^2 w}{\partial y^2} + 2B_{16}\frac{\partial^2 w}{\partial x\,\partial y}\right) +$$
$$+\frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\Big[(B_{11}a_{55} + B_{16}a_{45})\frac{\partial \varphi_0}{\partial x} + (B_{12}a_{45} + B_{16}a_{55})\frac{\partial \varphi_0}{\partial y} +$$
$$+(B_{12}a_{44} + B_{16}a_{45})\frac{\partial \psi_0}{\partial y} + (B_{11}a_{45} + B_{16}a_{44})\frac{\partial \psi_0}{\partial x}\Big] -$$
$$-A_1\left[Z_1 + \left(\frac{3}{2}\frac{z}{h} - 2\frac{z^3}{h^3}\right)Z_2\right], \tag{6.8}$$

$$\sigma_y = -z\left(B_{22}\frac{\partial^2 w}{\partial y^2} + B_{12}\frac{\partial^2 w}{\partial x^2} + 2B_{26}\frac{\partial^2 w}{\partial x\,\partial y}\right) +$$
$$+\frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\Big[(B_{22}a_{44} + B_{26}a_{45})\frac{\partial \psi_0}{\partial y} + (B_{12}a_{45} + B_{26}a_{44})\frac{\partial \psi_0}{\partial x} +$$
$$+(B_{12}a_{55} + B_{26}a_{45})\frac{\partial \varphi_0}{\partial x} + (B_{22}a_{45} + B_{26}a_{55})\frac{\partial \varphi_0}{\partial y}\Big] -$$
$$-A_2\left[Z_1 + \left(\frac{3}{2}\frac{z}{h} - 2\frac{z^3}{h^3}\right)Z_2\right], \tag{6.9}$$

$$\tau_{xy} = -z\left(2B_{66}\frac{\partial^2 w}{\partial x\,\partial y} + B_{16}\frac{\partial^2 w}{\partial x^2} + B_{26}\frac{\partial^2 w}{\partial y^2}\right) +$$
$$+\frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\Big[(B_{66}a_{55} + B_{26}a_{45})\frac{\partial \varphi_0}{\partial y} + (B_{66}a_{45} + B_{16}a_{55})\frac{\partial \varphi_0}{\partial x} +$$
$$+(B_{66}a_{44} + B_{16}a_{45})\frac{\partial \psi_0}{\partial x} + (B_{66}a_{45} + B_{26}a_{44})\frac{\partial \psi_0}{\partial y}\Big] -$$
$$-A_3\left[Z_1 + \left(\frac{3}{2}\frac{z}{h} - 2\frac{z^3}{h^3}\right)Z_2\right]. \tag{6.10}$$

Substituting expressions (6.8) – (6.10) into (2.38) – (2.40) we obtain the following relationships for the bending and twisting moment resultants:

$$M_x = -\left(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2} + 2D_{16}\frac{\partial^2 w}{\partial x\,\partial y}\right) + \\ + \frac{h^2}{10}\left[(D_{11}a_{55} + D_{16}a_{45})\frac{\partial \varphi_0}{\partial x} + (D_{12}a_{45} + D_{16}a_{55})\frac{\partial \varphi_0}{\partial y} + \right. \\ \left. + (D_{12}a_{44} + D_{16}a_{45})\frac{\partial \psi_0}{\partial y} + (D_{11}a_{45} + D_{16}a_{44})\frac{\partial \psi_0}{\partial x}\right] - \frac{h^2}{10}A_1 Z_2, \quad (6.11)$$

$$M_y = -\left(D_{22}\frac{\partial^2 w}{\partial y^2} + D_{12}\frac{\partial^2 w}{\partial x^2} + 2D_{26}\frac{\partial^2 w}{\partial x\,\partial y}\right) + \\ + \frac{h^2}{10}\left[(D_{22}a_{44} + D_{26}a_{45})\frac{\partial \psi_0}{\partial y} + (D_{12}a_{45} + D_{26}a_{44})\frac{\partial \psi_0}{\partial x} + \right. \\ \left. + (D_{12}a_{55} + D_{26}a_{45})\frac{\partial \varphi_0}{\partial x} + (D_{22}a_{45} + D_{26}a_{55})\frac{\partial \varphi_0}{\partial y}\right] - \frac{h^2}{10}A_2 Z_2, \quad (6.12)$$

$$H = -\left(D_{16}\frac{\partial^2 w}{\partial x^2} + D_{26}\frac{\partial^2 w}{\partial y^2} + 2D_{66}\frac{\partial^2 w}{\partial x\,\partial y}\right) + \\ + \frac{h^2}{10}\left[(D_{66}a_{55} + D_{26}a_{45})\frac{\partial \varphi_0}{\partial y} + (D_{66}a_{45} + D_{16}a_{55})\frac{\partial \varphi_0}{\partial x} + \right. \\ \left. + (D_{66}a_{44} + D_{16}a_{45})\frac{\partial \psi_0}{\partial x} + (D_{66}a_{45} + D_{26}a_{44})\frac{\partial \psi_0}{\partial y}\right] - \frac{h^2}{10}A_3 Z_2. \quad (6.13)$$

Transverse shearing stress resultants can be determined from equations (2.48) and (2.49). Substitution of values of the moments (6.11) – (6.13) into (2.48) and (2.49) we have then for these transverse shearing stress resultants:

$$N_x = -\left[D_{11}\frac{\partial^3 w}{\partial x^3} + 3D_{16}\frac{\partial^3 w}{\partial x^2\,\partial y} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x\,\partial y^2} + D_{26}\frac{\partial^3 w}{\partial y^3}\right] + \\ + \frac{h^2}{10}\left\{(D_{11}a_{55} + D_{16}a_{45})\frac{\partial^2}{\partial x^2} + \right. \\ \left. + [a_{45}(D_{12} + D_{66}) + 2a_{55}D_{16}]\frac{\partial^2}{\partial x\,\partial y} + (D_{66}a_{55} + D_{26}a_{45})\frac{\partial^2}{\partial y^2}\right\}\varphi_0 + \\ + \frac{h^2}{10}\left\{(D_{11}a_{45} + D_{16}a_{44})\frac{\partial^2}{\partial x^2} + [a_{44}(D_{12} + D_{66}) + 2a_{45}D_{16}]\frac{\partial^2}{\partial x\,\partial y} + \right. \\ \left. + (D_{66}a_{45} + D_{26}a_{44})\frac{\partial^2}{\partial y^2}\right\}\psi_0 - \frac{h^2}{10}\left(A_1\frac{\partial Z_2}{\partial x} + A_3\frac{\partial Z_2}{\partial y}\right), \quad (6.14)$$

$$N_y = -\left[D_{22}\frac{\partial^3 w}{\partial y^3} + 3D_{26}\frac{\partial^3 w}{\partial y^2\,\partial x} + (D_{12}+2D_{66})\frac{\partial^3 w}{\partial y\,\partial x^2} + D_{16}\frac{\partial^3 w}{\partial x^3}\right] +$$
$$+\frac{h^2}{10}\Big\{(D_{22}a_{44}+D_{26}a_{45})\frac{\partial^2}{\partial y^2} +$$
$$+[a_{45}(D_{12}+D_{66})+2a_{44}D_{26}]\frac{\partial^2}{\partial x\,\partial y} + (D_{66}a_{44}+D_{16}a_{45})\frac{\partial^2}{\partial x^2}\Big\}\psi_0 +$$
$$+\frac{h^2}{10}\Big\{(D_{22}a_{45}+D_{26}a_{55})\frac{\partial^2}{\partial y^2} + [a_{55}(D_{12}+D_{66})+2a_{45}D_{26}]\frac{\partial^2}{\partial x\,\partial y} +$$
$$+(D_{66}a_{45}+D_{16}a_{55})\frac{\partial^2}{\partial x^2}\Big\}\varphi_0 - \frac{h^2}{10}\left(A_2\frac{\partial Z_2}{\partial y} + A_3\frac{\partial Z_2}{\partial x}\right). \qquad (6.15)$$

Finally, substituting (6.14) and (6.15) for N_x and N_y into (2.47) a differential equation of fourth order with respect to one undetermined function is obtained:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 4D_{16}\frac{\partial^4 w}{\partial x^3\,\partial y} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\,\partial y^2} +$$
$$+4D_{26}\frac{\partial^4 w}{\partial x\,\partial y^3} + D_{22}\frac{\partial^4 w}{\partial y^4} = Z_2 + \frac{h^2}{10}\Big\{(D_{11}a_{55}+D_{16}a_{45})\frac{\partial^3}{\partial x^3} +$$
$$+[a_{45}(D_{12}+2D_{66})+3a_{55}D_{16}]\frac{\partial^3}{\partial x^2\,\partial y} +$$
$$+[a_{55}(D_{12}+2D_{66})+3a_{45}D_{26}]\frac{\partial^3}{\partial x\,\partial y^2} + (D_{22}a_{45}+D_{26}a_{55})\frac{\partial^3}{\partial y^3}\Big\}\varphi_0 +$$
$$+\frac{h^2}{10}\Big\{(D_{22}a_{44}+D_{26}a_{45})\frac{\partial^3}{\partial y^3} + [a_{45}(D_{12}+2D_{66})+3a_{44}D_{26}]\frac{\partial^3}{\partial y^2\,\partial x} +$$
$$+[a_{44}(D_{12}+2D_{66})+3a_{45}D_{16}]\frac{\partial^3}{\partial y\,\partial x^2} + (D_{11}a_{45}+D_{16}a_{44})\frac{\partial^3}{\partial x^3}\Big\}\psi_0 -$$
$$-\frac{h^2}{10}\left[A_1\frac{\partial^2}{\partial x^2} + 2A_3\frac{\partial^2}{\partial x\,\partial y} + A_2\frac{\partial^2}{\partial y^2}\right]Z_2. \qquad (6.16)$$

The undetermined function w(x, y) should satisfy not only the differential equation (6.16), for which the left side is the same as the corresponding left side of the equation of the classical theory, but also the boundary conditions at the plate edge which, as a rule, have the same form as the boundary conditions of the classical theory.

Boundary conditions for five principal cases are given below [31]. The edge for the general case is considered as being curvilinear with normal n and contour arc s.

(a) Free edge

$$M_n = 0, \qquad N_n + \frac{\partial H_{nt}}{\partial s} = 0. \tag{6.17}$$

(b) Simply-supported edge

$$w = 0, \qquad M_n = 0. \tag{6.18}$$

(c) Rigidly fixed edge

$$w = 0, \qquad u_n|_{z=\pm z_0} = 0. \tag{6.19}$$

(d) Loaded edge

$$M_n = M_n^*, \qquad N_n + \frac{\partial H_{nt}}{\partial s} = N_n^*. \tag{6.20}$$

(e) Displaced edge

$$w = w^*, \qquad u_n|_{z=\pm z_0} = u_n^*. \tag{6.21}$$

Let us indicate for completeness that the given boundary conditions may be expanded to some extent (see the examples in the next Section) for this theory.

If the plate under consideration is made of orthotropic material and the directions of the x, y and z axes coincide with the principal directions of elasticity, then expressions (5.1) and (5.2) are valid. Therefore, from (6.8) – (6.15) we have the following results for stresses and internal stress and moment resultants:

$$\sigma_x = -z\left(B_{11}\frac{\partial^2 w}{\partial x^2} + B_{12}\frac{\partial^2 w}{\partial y^2}\right) + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\times$$
$$\times\left(B_{11}a_{55}\frac{\partial \varphi_0}{\partial x} + B_{12}a_{44}\frac{\partial \psi_0}{\partial y}\right) - A_1\left[Z_1 + \left(\frac{3}{2}\frac{z}{h} - 2\frac{z^3}{h^3}\right)Z_2\right], \tag{6.22}$$

$$\sigma_y = -z\left(B_{22}\frac{\partial^2 w}{\partial y^2} + B_{12}\frac{\partial^2 w}{\partial x^2}\right) + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\times$$
$$\times\left(B_{22}a_{44}\frac{\partial \psi_0}{\partial y} + B_{12}a_{55}\frac{\partial \varphi_0}{\partial x}\right) - A_2\left[Z_1 + \left(\frac{3}{2}\frac{z}{h} - 2\frac{z^3}{h^3}\right)Z_2\right], \tag{6.23}$$

$$\tau_{xy} = -2zB_{66}\frac{\partial^2 w}{\partial x\,\partial y} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(B_{66}a_{55}\frac{\partial \varphi_0}{\partial y} + B_{66}a_{44}\frac{\partial \psi_0}{\partial x}\right), \tag{6.24}$$

$$M_x = -\left(D_{11}\frac{\partial^2 w}{\partial x^2} + D_{12}\frac{\partial^2 w}{\partial y^2}\right) - \frac{h^2}{10}A_1 Z_2 +$$
$$+ \frac{h^2}{10}\left(D_{11}a_{55}\frac{\partial \varphi_0}{\partial x} + D_{12}a_{44}\frac{\partial \psi_0}{\partial y}\right), \tag{6.25}$$

$$M_y = -\left(D_{22}\frac{\partial^2 w}{\partial y^2} + D_{12}\frac{\partial^2 w}{\partial x^2}\right) - \frac{h^2}{10}A_2 Z_2 + \\ + \frac{h^2}{10}\left(D_{22}a_{44}\frac{\partial \psi_0}{\partial y} + D_{12}a_{55}\frac{\partial \varphi_0}{\partial x}\right), \quad (6.26)$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial x\,\partial y} + \frac{h^2}{10}D_{66}\left(a_{55}\frac{\partial \varphi_0}{\partial y} + a_{44}\frac{\partial \psi_0}{\partial x}\right), \quad (6.27)$$

$$N_x = -\left[D_{11}\frac{\partial^3 w}{\partial x^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x\,\partial y^2}\right] - \frac{h^2}{10}A_1\frac{\partial Z_2}{\partial x} + \\ + \frac{h^2}{10}\left[a_{55}\left(D_{11}\frac{\partial^2 \varphi_0}{\partial x^2} + D_{66}\frac{\partial^2 \varphi_0}{\partial y^2}\right) + a_{44}(D_{12} + D_{66})\frac{\partial^2 \psi_0}{\partial x\,\partial y}\right] \quad (6.28)$$

$$N_y = -\left[D_{22}\frac{\partial^3 w}{\partial y^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x\,\partial y^2}\right] - \frac{h^2}{10}A_2\frac{\partial Z_2}{\partial y} + \\ + \frac{h^2}{10}\left[a_{44}\left(D_{22}\frac{\partial^2 \psi_0}{\partial y^2} + D_{66}\frac{\partial^2 \psi_0}{\partial x^2}\right) + a_{55}(D_{12} + D_{66})\frac{\partial^2 \varphi_0}{\partial x\,\partial y}\right] \quad (6.29)$$

Finally, from equation (6.16) the specialized differential equation for this problem is obtained:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2\,\partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} = \\ = Z_2 + \frac{h^2}{10}a_{55}\left[D_{11}\frac{\partial^3}{\partial x^3} + (D_{12} + 2D_{66})\frac{\partial^3}{\partial x\,\partial y^2}\right]\varphi_0 + \\ + \frac{h^2}{10}a_{44}\left[D_{22}\frac{\partial^3}{\partial y^3} + (D_{12} + 2D_{66})\frac{\partial^3}{\partial y\,\partial x^2}\right]\psi_0 - \\ - \frac{h^2}{10}\left(A_1\frac{\partial^2}{\partial x^2} + A_2\frac{\partial^2}{\partial y^2}\right)Z_2. \quad (6.30)$$

In regard to the assumed functions $\varphi_0(x, y)$ and $\psi_0(x, y)$, for the orthotropic plate they are

$$\varphi_0 = -\left[B_{11}\frac{\partial^3 w_0}{\partial x^3} + (B_{12} + 2B_{66})\frac{\partial^3 w_0}{\partial x\,\partial y^2}\right], \quad (6.31)$$

$$\psi_0 = -\left[B_{22}\frac{\partial^3 w_0}{\partial y^3} + (B_{12} + 2B_{66})\frac{\partial^3 w_0}{\partial y\,\partial x^2}\right], \quad (6.32)$$

where w_0 is the normal displacement determined from the solution of the classical problem, i.e., from the equation

$$D_{11}\frac{\partial^4 w_0}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w_0}{\partial x^2\,\partial y^2}+D_{22}\frac{\partial^4 w_0}{\partial y^4}=Z_2 \qquad (6.33)$$

and satisfying the corresponding boundary conditions.

Thus the final expressions for the stress identities and the differential equation are obtained by substituting values of φ_0 and ψ_0 from (6.31) and (6.32) into (6.22) – (6.29) and (6.30). The resulting differential equation

$$\begin{aligned}D_{11}\frac{\partial^4 w}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x^2\,\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}=\\ =Z_2-\frac{h^5}{120}\left\{a_{55}\frac{\partial^2}{\partial x^2}\left[B_{11}^2\frac{\partial^4 w_0}{\partial x^4}+2B_{11}(B_{12}+2B_{66})\frac{\partial^4 w_0}{\partial x^2\,\partial y^2}+\right.\right.\\ \left.+(B_{12}+2B_{66})^2\frac{\partial^4 w_0}{\partial y^4}\right]+a_{44}\frac{\partial^2}{\partial y^2}\left[B_{22}^2\frac{\partial^4 w_0}{\partial y^4}+\right.\\ \left.\left.+2B_{22}(B_{12}+2B_{66})\frac{\partial^4 w_0}{\partial x^2\,\partial y^2}+(B_{12}+2B_{66})^2\frac{\partial^4 w_0}{\partial x^4}\right]\right\}-\\ -\frac{h^2}{10}\left(A_1\frac{\partial^2 Z_2}{\partial x^2}+A_2\frac{\partial^2 Z_2}{\partial y^2}\right) \qquad (6.34)\end{aligned}$$

is given here without giving all the cumbersome intermediate equations.

For the given case of the orthotropic plate the boundary conditions remain unchanged, i.e., the conditions (6.17) – (6.21) are valid.

It can be noted in conclusion that the equations of the theory suggested here are in fact equivalent to equations of the first three approximations of the principal iterational process of A. L. Gol'denveizer [32], which is used to determine slowly changing states of stress of the plate. Generally speaking, this process is applied to the entire plate and produces more accurate values of stresses and displacements at some distance from the line of distortion.

The state of stress near rapid distortions are excluded from consideration in this particular theory and only the influence on the principal state of stress is considered, as opposed to the general theory with which it is possible to derive formally not only the principal state of stress but the boundary stress conditions as well.

Finally, assuming that $A_1 = 0$ and $A_2 = 0$ in the equations of this section we obtain all the equations of that improved theory of anisotropic plates which disregards the effect of the stress σ_z on the stressed and deformed state of the plate.

7. Theory of Bending of Plates with Cylindrical Anisotropy

A general and a particular theory of bending of plates with cylindrical anisotropy can be obtained in the manner discussed in the preceding sections.

It is assumed that a plate with cylindrical anisotropy is specially orthotropic and all radial planes which pass through the axis of anisotropy are planes of elastic symmetry [33]. In addition, the pole of anisotropy, a point where the axis of anisotropy intersects the plate middle plane (which is assumed to be perpendicular to this axis), is the origin of the cylindrical system of coordinates $r\theta z$ (see Section 2, Chapter I), and the z-axis is directed along the axis of anisotropy (Figure 2.).

Finally, we consider the case where $f_i(z)$ changes according to a parabolic law (4.1) and the plate is loaded with only normally applied load $Z^{\pm}(r, \theta) \neq 0$, $X^{\pm} = 0$, $Y^{\pm} = 0$.

1. General theory. The general theory is derived on the basis of the initial assumptions (a) and (b) given in Section 1 of this chapter and the basic equations and identities of the three-dimensional theory of elasticity of cylindrically anisotropic bodies.

Using assumptions (a) and (b) it is assumed that

$$e_z = 0, \tag{7.1}$$

$$\tau_{rz} = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right)\varphi, \quad \tau_{\theta z} = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right)\psi, \tag{7.2}$$

where $\varphi(r, \theta)$ and $\psi(r, \theta)$ are arbitrary undetermined functions of the coordinates r, θ.

The method for the derivation of the equations is the same as in Sections 2 – 5. Therefore, in many instances only results are given without detailed explanations.

Due to assumptions (7.1), (7.2) and (1.2.5) and considering (1.4.15) we obtain for displacements:

$$u_z = u_z(r, \vartheta) = w(r, \vartheta), \tag{7.3}$$

$$u_r = u - z\frac{\partial w}{\partial r} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)a_r\varphi, \tag{7.4}$$

$$u_\vartheta = v - z\frac{\partial w}{r\partial\vartheta} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)a_\vartheta\psi; \tag{7.5}$$

where $w(r, \theta)$ is the normal displacement of the middle plane; $u = u(r, \theta)$ and $v = v(r, \theta)$ are the tangential displacements of the middle plane along coordinates r and θ, respectively. On the basis of (1.2.5) and using (7.3) – (7.5) the following expressions for stresses are obtained, after substituting the strains e_r, e_θ, $e_{r\theta}$ into the generalized Hooke's law (1.4.15):

$$\sigma_r = B_r\left(\frac{\partial u}{\partial r} - z\frac{\partial^2 w}{\partial r^2}\right) + B_r\nu_\vartheta\left[\frac{1}{r}\frac{\partial v}{\partial \vartheta} + \frac{u}{r} - z\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial \vartheta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\right] +$$
$$+ \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left[B_r a_r\left(\frac{\partial \varphi}{\partial r} + \nu_\vartheta\frac{\varphi}{r}\right) + B_r\nu_\vartheta a_\vartheta\frac{1}{r}\frac{\partial \psi}{\partial \vartheta}\right] -$$
$$- A_1\left[Z_1 + \left(\frac{3}{2}\frac{z}{h} - 2\frac{z^3}{h^3}\right)Z_2\right], \qquad (7.6)$$

$$\sigma_\vartheta = B_\vartheta\left[\frac{1}{r}\frac{\partial v}{\partial \vartheta} + \frac{u}{r} - z\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial \vartheta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right)\right] +$$
$$+ B_\vartheta\nu_r\left(\frac{\partial u}{\partial r} - z\frac{\partial^2 w}{\partial r^2}\right) + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left[B_\vartheta a_\vartheta\frac{1}{r}\frac{\partial \psi}{\partial \vartheta} +\right.$$
$$\left. + B_\vartheta a_r\left(\frac{\varphi}{r} + \nu_r\frac{\partial \varphi}{\partial r}\right)\right] - A_2\left[Z_1 + \left(\frac{3}{2}\frac{z}{h} - 2\frac{z^3}{h^3}\right)Z_2\right], \qquad (7.7)$$

$$\tau_{\vartheta r} = B_k\left\{\frac{1}{r}\frac{\partial u}{\partial \vartheta} + \frac{\partial v}{\partial r} - \frac{v}{r} - 2z\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \vartheta}\right) +\right.$$
$$\left. + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left[a_r\frac{1}{r}\frac{\partial \varphi}{\partial \vartheta} + a_\vartheta\left(\frac{\partial \psi}{\partial r} - \frac{\psi}{r}\right)\right]\right\}, \qquad (7.8)$$

where (see (1.4.15))

$$\left.\begin{aligned} B_r &= \frac{E_r}{1 - \nu_r\nu_\vartheta}, \quad B_\vartheta = \frac{E_\vartheta}{1 - \nu_r\nu_\vartheta}, \quad B_k = G_{r\vartheta}, \\ a_r &= \frac{1}{B_{rz}} = \frac{1}{G_{rz}}, \quad a_\vartheta = \frac{1}{B_{\vartheta z}} = \frac{1}{G_{\vartheta z}}, \\ A_1 &= -\frac{E_r}{E_z}\frac{\nu_{rz} + \nu_\vartheta\nu_{\vartheta z}}{1 - \nu_r\nu_\vartheta}, \quad A_2 = -\frac{E_\vartheta}{E_z}\frac{\nu_{\vartheta z} + \nu_r\nu_{rz}}{1 - \nu_r\nu_\vartheta}, \end{aligned}\right\} \qquad (7.9)$$

$$E_r\nu_{\vartheta r} = E_\vartheta\nu_{r\vartheta}, \quad \nu_{\vartheta r} = \nu_r, \quad \nu_{r\vartheta} = \nu_\vartheta.$$

The last terms in equations (7.6) and (7.7) appear due to the stress σ_z which is determined as before from (1.2.6) and the conditions at the surface (2.25).

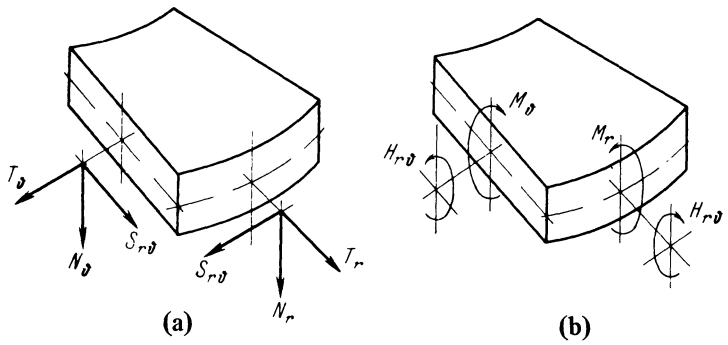

Figure 9.

Internal stress and moment resultants (Figure 9.) which are obtained from the given stresses are:

$$T_r = C_r \frac{\partial u}{\partial r} + C_r \nu_\vartheta \frac{1}{r}\left(\frac{\partial v}{\partial \vartheta} + u\right) - hA_1 Z_1, \tag{7.10}$$

$$T_\vartheta = C_\vartheta \frac{1}{r}\left(\frac{\partial v}{\partial \vartheta} + u\right) + C_\vartheta \nu_r \frac{\partial u}{\partial r} - hA_2 Z_1, \tag{7.11}$$

$$S_{\vartheta r} = C_k \left[\frac{1}{r}\frac{\partial u}{\partial \vartheta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right], \tag{7.12}$$

$$M_r = -D_r \frac{\partial^2 w}{\partial r^2} - D_r \nu_\vartheta \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial \vartheta^2} + \frac{\partial w}{\partial r}\right) + \\ + \frac{h^2}{10}\left[D_r a_r \left(\frac{\partial \varphi}{\partial r} + \nu_\vartheta \frac{\varphi}{r}\right) + D_r \nu_\vartheta a_\vartheta \frac{1}{r}\frac{\partial \psi}{\partial \vartheta}\right] - \frac{h^2}{10} A_1 Z_2, \tag{7.13}$$

$$M_\vartheta = -D_\vartheta \frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial \vartheta^2} + \frac{\partial w}{\partial r}\right) - D_\vartheta \nu_r \frac{\partial^2 w}{\partial r^2} + \\ + \frac{h^2}{10}\left[D_\vartheta a_\vartheta \frac{1}{r}\frac{\partial \psi}{\partial \vartheta} + D_\vartheta a_r \left(\frac{\varphi}{r} + \nu_r \frac{\partial \varphi}{\partial r}\right)\right] - \frac{h^2}{10} A_2 Z_2, \tag{7.14}$$

$$H_{r\vartheta} = D_k \left\{-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \vartheta}\right) + \frac{h^2}{10}\left[a_r \frac{1}{r}\frac{\partial \varphi}{\partial \vartheta} + a_\vartheta \left(\frac{\partial \psi}{\partial r} - \frac{\psi}{r}\right)\right]\right\}, \tag{7.15}$$

$$N_r = \frac{h^3}{12}\varphi, \qquad N_\vartheta = \frac{h^3}{12}\psi, \tag{7.16}$$

where D_r, D_θ and C_r, C_θ are bending stiffnesses and tension-compression stiffnesses for the directions r and θ, respectively; D_k is the torsional rigidity; C_k is the shear rigidity. These values are expressed in the following manner:

$$\left.\begin{aligned} D_r &= \frac{h^3}{12}B_r, \quad D_\vartheta = \frac{h^3}{12}B_\vartheta, \quad D_k = \frac{h^3}{12}B_k, \\ C_r &= hB_r, \quad C_\vartheta = hB_\vartheta, \quad C_k = hB_k, \\ D_{r\vartheta} &= D_r\nu_\vartheta + 2D_k, \end{aligned}\right\} \tag{7.17}$$

and Z_1 and Z_2 are as given in (2.29).

In a manner identical to that used in Section 2, the following system of differential equations of plate equilibrium for the case $X^{\pm} = 0$, $Y^{\pm} = 0$, $Z^{\pm} \neq 0$ is obtained from (1.2.6), including consideration of the surface conditions:

$$\frac{\partial T_r}{\partial r} + \frac{1}{r}\frac{\partial S_{\vartheta r}}{\partial \vartheta} + \frac{T_r - T_\vartheta}{r} = 0, \tag{7.18}$$

$$\frac{\partial S_{\vartheta r}}{\partial r} + \frac{1}{r}\frac{\partial T_\vartheta}{\partial \vartheta} + \frac{2}{r}S_{\vartheta r} = 0, \tag{7.19}$$

$$\frac{\partial N_r}{\partial r} + \frac{1}{r}\frac{\partial N_\vartheta}{\partial \vartheta} + \frac{1}{r}N_r = -Z_2, \tag{7.20}$$

$$\frac{\partial M_r}{\partial r} + \frac{1}{r}\frac{\partial H_{\vartheta r}}{\partial \vartheta} + \frac{M_r - M_\vartheta}{r} = N_r, \tag{7.21}$$

$$\frac{\partial H_{\vartheta r}}{\partial r} + \frac{1}{r}\frac{\partial M_\vartheta}{\partial \vartheta} + \frac{2}{r}H_{\vartheta r} = N_\vartheta. \tag{7.22}$$

Substituting the values of the internal stress and moment resultants into the equations of equilibrium, a system of five differential equations with respect to five undetermined functions is obtained:

$$C_r\left(r\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\right) + C_k\frac{1}{r}\frac{\partial^2 u}{\partial \vartheta^2} - C_\vartheta\frac{u}{r} + (C_\vartheta\nu_r + C_k)\frac{\partial^2 v}{\partial r\,\partial \vartheta} - \\ - (C_\vartheta + C_k)\frac{1}{r}\frac{\partial v}{\partial \vartheta} = hA_1\frac{\partial}{\partial r}(rZ_1) - hA_2Z_1, \tag{7.23}$$

$$C_\vartheta\frac{1}{r}\frac{\partial^2 v}{\partial \vartheta^2} + C_k\left(r\frac{\partial^2 v}{\partial r^2} + \frac{\partial v}{\partial r} - \frac{v}{r}\right) + \\ + (C_r\nu_\vartheta + C_k)\frac{\partial^2 u}{\partial r\,\partial \vartheta} + (C_\vartheta + C_k)\frac{1}{r}\frac{\partial u}{\partial \vartheta} = hA_2\frac{\partial Z_1}{\partial \vartheta}, \tag{7.24}$$

$$\frac{1}{r}\frac{\partial(r\varphi)}{\partial r}+\frac{1}{r}\frac{\partial\psi}{\partial\vartheta}=-\frac{12}{h^3}Z_2, \tag{7.25}$$

$$D_r\frac{\partial}{\partial r}\left(r\frac{\partial^2 w}{\partial r^2}\right)+D_{r\vartheta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\vartheta^2}\right)-D_\vartheta\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\vartheta^2}\right)-$$
$$-\frac{h^2}{10}\left\{a_r\left[D_r\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)+D_k\frac{1}{r}\frac{\partial^2\varphi}{\partial\vartheta^2}-D_\vartheta\frac{\varphi}{r}\right]+\right.$$
$$\left.+a_\vartheta\left[(D_\vartheta\nu_r+D_k)\frac{\partial^2\psi}{\partial r\,\partial\vartheta}-(D_\vartheta+D_k)\frac{1}{r}\frac{\partial\psi}{\partial\vartheta}\right]\right\}+$$
$$+\frac{h^3}{12}r\varphi=A_2\frac{h^2}{10}Z_2-A_1\frac{h^2}{10}\frac{\partial}{\partial r}(rZ_2), \tag{7.26}$$

$$D_\vartheta\left(\frac{1}{r^2}\frac{\partial^3 w}{\partial\vartheta^3}+\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\vartheta}\right)+D_{r\vartheta}\frac{\partial^3 w}{\partial r^2\,\partial\vartheta}-$$
$$-\frac{h^2}{10}\left\{a_\vartheta\left[D_\vartheta\frac{1}{r}\frac{\partial^2\psi}{\partial\vartheta}+D_k\left(r\frac{\partial^2\psi}{\partial r^2}+\frac{\partial\psi}{\partial r}-\frac{\psi}{r}\right)\right]+\right.$$
$$\left.+a_r\left[(D_r\nu_\vartheta+D_k)\frac{\partial^2\varphi}{\partial r\,\partial\vartheta}+(D_\vartheta+D_k)\frac{1}{r}\frac{\partial\varphi}{\partial\vartheta}\right]\right\}+$$
$$+\frac{h^3}{12}r\psi=-A_2\frac{h^2}{10}\frac{\partial Z_2}{\partial\vartheta}. \tag{7.27}$$

The boundary conditions do not change, i.e., we have equalities (3.19) – (3.21) and the corresponding non-uniform boundary conditions.

2. Particular theory. The particular theory is based on the initial assumptions (a) and (b) given in Section 6 of this chapter, on the basic equations and identities of the three-dimensional theory of elasticity, and on some results of the classical theory of bending of anisotropic plates [34].

The order of derivation of the equations is similar to that given in Section 6. Therefore, only the final results are given here.

The basic expressions for stresses are:

$$\sigma_r=-z\left[B_r\frac{\partial^2 w}{\partial r^2}+B_r\nu_\vartheta\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial\vartheta^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)\right]+$$
$$+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\left[B_r a_r\left(\frac{\partial\varphi_0}{\partial r}+\nu_\vartheta\frac{\varphi_0}{r}\right)+B_r\nu_\vartheta a_\vartheta\frac{1}{r}\frac{\partial\psi_0}{\partial\vartheta}\right]-$$
$$-A_1\left[Z_1+\left(\frac{3}{2}\frac{z}{h}-2\frac{z^3}{h^3}\right)Z_2\right], \tag{7.28}$$

$$\sigma_\vartheta = -z\left[B_\vartheta\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial\vartheta^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+B_\vartheta\nu_r\frac{\partial^2 w}{\partial r^2}\right]+$$
$$+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\left[B_\vartheta a_\vartheta\frac{1}{r}\frac{\partial\psi_0}{\partial\vartheta}+B_\vartheta a_r\left(\frac{\varphi_0}{r}+\nu_r\frac{\partial\varphi_0}{\partial r}\right)\right]-$$
$$-A_2\left[Z_1+\left(\frac{3}{2}\frac{z}{h}-2\frac{z^3}{h^3}\right)Z_2\right], \qquad (7.29)$$

$$\tau_{\vartheta r} = -2zB_h\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\vartheta}\right)+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\left[a_r\frac{1}{r}\frac{\partial\varphi_0}{\partial\vartheta}+a_\vartheta\left(\frac{\partial\psi_0}{\partial r}-\frac{\psi_0}{r}\right)\right] \qquad (7.30)$$

The bending and torsional moment resultants are:

$$M_r = -D_r\frac{\partial^2 w}{\partial r^2}-D_r\nu_\vartheta\frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\vartheta^2}+\frac{\partial w}{\partial r}\right)+$$
$$+\frac{h^2}{10}\left[D_r a_r\left(\frac{\partial\varphi_0}{\partial r}+\nu_\vartheta\frac{\varphi_0}{r}\right)+D_r\nu_\vartheta a_\vartheta\frac{1}{r}\frac{\partial\psi_0}{\partial\vartheta}\right]-\frac{h^2}{10}A_1Z_2, \qquad (7.31)$$

$$M_\vartheta = -D_\vartheta\frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\vartheta^2}+\frac{\partial w}{\partial r}\right)-D_\vartheta\nu_r\frac{\partial^2 w}{\partial r^2}+$$
$$+\frac{h^2}{10}\left[D_\vartheta a_\vartheta\frac{1}{r}\frac{\partial\psi_0}{\partial\vartheta}+D_\vartheta a_r\left(\frac{\varphi_0}{r}+\nu_r\frac{\partial\varphi_0}{\partial r}\right)\right]-\frac{h^2}{10}A_2Z_2, \qquad (7.32)$$

$$H_{r\vartheta} = D_k\left\{-2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\vartheta}\right)+\frac{h^2}{10}\left[a_r\frac{\partial\varphi_0}{r\,\partial\vartheta}+a_\vartheta\left(\frac{\partial\psi_0}{\partial r}-\frac{\psi_0}{r}\right)\right]\right\} \qquad (7.33)$$

The transverse shearing stress resultants are:

$$N_r = -D_r\left(\frac{\partial^3 w}{\partial r^3}+\frac{1}{r}\frac{\partial^2 w}{\partial r^2}\right)-D_{r\vartheta}\frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2}\left(\frac{\partial w}{\partial r}-\frac{w}{r}\right)+$$
$$+D_\vartheta\frac{1}{r^2}\left(\frac{\partial w}{\partial r}+\frac{1}{r}\frac{\partial^2 w}{\partial\vartheta^2}\right)+\frac{h^2}{10}\left\{a_r\left[D_r\left(\frac{\partial^2\varphi_0}{\partial r^2}+\frac{1+\nu_\vartheta}{r}\frac{\partial\varphi_0}{\partial r}\right)+\right.\right.$$
$$\left.+D_k\frac{1}{r^2}\frac{\partial^2\varphi_0}{\partial\vartheta^2}-D_\vartheta\left(\frac{\varphi_0}{r^2}+\nu_r\frac{1}{r}\frac{\partial\varphi_0}{\partial r}\right)\right]+$$
$$\left.+a_\vartheta\left[D_r\nu_\vartheta\frac{1}{r}\frac{\partial^2\psi_0}{\partial r\,\partial\vartheta}+D_k\left(\frac{1}{r}\frac{\partial^2\psi_0}{\partial r\,\partial\vartheta}-\frac{1}{r^2}\frac{\partial\psi_0}{\partial\vartheta}\right)-D_\vartheta\frac{1}{r^2}\frac{\partial\psi_0}{\partial\vartheta}\right]\right\}-$$
$$-\frac{h^2}{10}\left[A_1\frac{\partial Z_2}{\partial r}-\frac{1}{r}(A_1-A_2)Z_2\right], \qquad (7.34)$$

$$N_\vartheta = -D_\vartheta\left(\frac{1}{r^3}\frac{\partial^3 w}{\partial\vartheta^3}+\frac{1}{r^2}\frac{\partial^2 w}{\partial r\,\partial\vartheta}\right)-D_{r\vartheta}\frac{1}{r}\frac{\partial^3 w}{\partial r^2\,\partial\vartheta}+$$

$$+\frac{h^2}{10}\left\{a_\vartheta\left[D_\vartheta\frac{1}{r^2}\frac{\partial^2\psi_0}{\partial\vartheta^2}+D_k\left(\frac{\partial^2\psi_0}{\partial r^2}+\frac{1}{r}\frac{\partial\psi_0}{\partial r}-\frac{\psi_0}{r^2}\right)\right]+\right.$$

$$+a_r\left[D_\vartheta\left(\frac{1}{r^2}\frac{\partial\varphi_0}{\partial\vartheta}+\nu_r\frac{1}{r}\frac{\partial^2\varphi_0}{\partial r\,\partial\vartheta}\right)+\right.$$

$$\left.\left.\left.+D_k\left(\frac{1}{r}\frac{\partial^2\varphi_0}{\partial r\,\partial\vartheta}+\frac{1}{r^2}\frac{\partial\varphi_0}{\partial\vartheta}\right)\right]\right\}-A_2\frac{1}{r}\frac{\partial Z_2}{\partial\vartheta}. \qquad (7.35)$$

And finally the basic differential equation of the problem is:

$$D_r\frac{\partial^4 w}{\partial r^4}+2D_{r\vartheta}\frac{1}{r^2}\frac{\partial^4 w}{\partial r^2\partial\vartheta^2}+D_\vartheta\frac{1}{r^4}\frac{\partial^4 w}{\partial\vartheta^4}+2D_r\frac{1}{r}\frac{\partial^3 w}{\partial r^3}-2D_{r\vartheta}\frac{1}{r^3}\frac{\partial^3 w}{\partial r\,\partial\vartheta^2}-$$

$$-D_\vartheta\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}+2(D_\vartheta+D_{r\vartheta})\frac{1}{r^4}\frac{\partial^2 w}{\partial\vartheta^2}+D_\vartheta\frac{1}{r^3}\frac{\partial w}{\partial r}=$$

$$=Z_2+\frac{h^2}{10}\left\{a_r\left[D_r\left(\frac{2}{r}\frac{\partial^2\varphi_0}{\partial r^2}+\frac{\partial^3\varphi_0}{\partial r^3}\right)+D_{r\vartheta}\frac{1}{r^2}\frac{\partial^3\varphi_0}{\partial r\,\partial\vartheta^2}+\right.\right.$$

$$+D_\vartheta\left(\frac{1}{r^3}\frac{\partial^2\varphi_0}{\partial\vartheta^2}-\frac{1}{r^2}\frac{\partial\varphi_0}{\partial r}+\frac{\varphi_0}{r^3}\right)\Big]+a_\vartheta\left[D_{r\vartheta}\frac{1}{r}\frac{\partial^3\psi_0}{\partial r^2\partial\vartheta}+\right.$$

$$\left.\left.+D_\vartheta\left(\frac{1}{r^3}\frac{\partial^3\psi_0}{\partial\vartheta^3}-\frac{1}{r^2}\frac{\partial^2\psi_0}{\partial r\,\partial\vartheta}+\frac{1}{r^3}\frac{\partial\psi_0}{\partial\vartheta}\right)\right]\right\}-$$

$$-\frac{h^2}{10}\left[(2A_1-A_2)\frac{\partial Z_2}{r\,\partial r}+A_1\frac{\partial^2 Z_2}{\partial r^2}+A_2\frac{\partial^2 Z_2}{r^2\partial\vartheta}\right] \qquad (7.36)$$

In all these equations the functions φ_0 and ψ_0 can be taken as follows:

$$\varphi_0=-\left[B_r\left(\frac{\partial^3 w_0}{\partial r^3}+\frac{1}{r}\frac{\partial^2 w_0}{\partial r^2}\right)+B_{\vartheta r}\frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2}\left(\frac{\partial w_0}{\partial r}-\frac{w_0}{r}\right)-\right.$$

$$\left.-B_\vartheta\frac{1}{r^2}\left(\frac{\partial w_0}{\partial r}+\frac{1}{r}\frac{\partial^2 w_0}{\partial\vartheta^2}\right)\right]. \qquad (7.37)$$

$$\psi_0=-\left[B_{\vartheta r}\frac{1}{r}\frac{\partial^3 w_0}{\partial r^2\partial\vartheta}+B_\vartheta\frac{1}{r^2}\frac{\partial}{\partial\vartheta}\left(\frac{\partial w_0}{\partial r}+\frac{1}{r}\frac{\partial^2 w_0}{\partial\vartheta^2}\right)\right]; \qquad (7.38)$$

where w_0 is the normal displacement of the corresponding anisotropic plate as determined from the classical theory, i.e., the solution of the following differential equation

$$D_r \frac{\partial^4 w_0}{\partial r^4} + 2D_{r\vartheta} \frac{1}{r^2} \frac{\partial^4 w_0}{\partial r^2 \partial \vartheta^2} + D_\vartheta \frac{1}{r^4} \frac{\partial^4 w_0}{\partial \vartheta^4} + 2D_r \frac{1}{r} \frac{\partial^3 w_0}{\partial r^3} -$$
$$- 2D_{r\vartheta} \frac{1}{r^3} \frac{\partial^3 w_0}{\partial r \partial \vartheta^2} - D_\vartheta \frac{1}{r^2} \frac{\partial^2 w_0}{\partial r^2} + 2(D_\vartheta + D_{r\vartheta}) \frac{1}{r^4} \frac{\partial^2 w_0}{\partial \vartheta^2} +$$
$$+ D_\vartheta \frac{1}{r^3} \frac{\partial w_0}{\partial r} = Z_2(r, \vartheta) \qquad (7.39)$$

with the corresponding boundary conditions.

All the basic boundary conditions remain unchanged, i.e., we have (6.17) – (6.21).

8. Geometrically Non-Linear Theories of Anisotropic Plates

A general and a particular theory are given here for the case that the plate displacements are of the same size as its thickness h. These theories are applied to specially orthotropic plates, i.e., plates made of orthotropic material with principal directions of elasticity coincident with the directions of coordinate lines.

These theories are presented on the assumption that not only are the elongations and shearing deformations small in comparison with unity, but the angles of rotation of the plate elements are also small with respect to unity [35].

As opposed to the non-linear theory of the very large deflections of plates, only those non-linear terms are preserved here from the basic equations of the non-linear theory of elasticity which are due to normal displacement w and its derivatives. Thus, instead of (1.3.1) – (1.3.4) we have the following expressions for the deformations:

$$\left.\begin{aligned} e_x &= \frac{\partial u_x}{\partial x} + \frac{1}{2}\left(\frac{\partial u_z}{\partial x}\right)^2, \\ e_y &= \frac{\partial u_y}{\partial y} + \frac{1}{2}\left(\frac{\partial u_z}{\partial y}\right)^2, \\ e_z &= \frac{\partial u_z}{\partial z} + \frac{1}{2}\left(\frac{\partial u_z}{\partial z}\right)^2, \end{aligned}\right\} \qquad (8.1)$$

$$\left.\begin{aligned} e_{xz} &= \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial z}, \\ e_{yz} &= \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} + \frac{\partial u_z}{\partial y}\frac{\partial u_z}{\partial z}, \\ e_{xy} &= \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial y}, \end{aligned}\right\} \tag{8.2}$$

and the equations of equilibrium without considering body forces are now:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0, \tag{8.3}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0, \tag{8.4}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u_z}{\partial x}\sigma_x + \frac{\partial u_z}{\partial y}\tau_{xy} + \tau_{xz}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u_z}{\partial x}\tau_{xy} + \frac{\partial u_z}{\partial y}\sigma_y + \tau_{yz}\right) + \\ + \frac{\partial}{\partial z}\left(\frac{\partial u_z}{\partial x}\tau_{xz} + \frac{\partial u_z}{\partial y}\tau_{yz} + \sigma_z\right) = 0. \tag{8.5}$$

Equations (1.3.5) and (1.3.6) can be simplified in an analogous manner.

1. General theory [36]. The general theory is based on the basic equations, (8.1) – (8.5) and the following assumptions:
   (a) a normal displacement u_z with respect to the middle plane does not depend on the z coordinate;
   (b) shearing stresses τ_{xz} and τ_{yz} or the corresponding deformations e_{xz} and e_{yz} change according to a given function $f_i(z)$ through the plate thickness;
   (c) the normal stress σ_z can be determined from the linearized equation of equilibrium (see formulas (2.24) – (2.25) and equation (8.5)).

   For simplicity we considered the case where $f_i(z)$ changes according to a parabolic law (4.1), and the plate is loaded only by normally applied load $Z^{\pm}(x, y) \neq 0$, $X^{\pm} = 0$, $Y^{\pm} = 0$.

   By accepting assumptions (a), (b) and (c) we assume in fact

$$\frac{\partial u_z}{\partial z} = 0, \qquad \sigma_z = Z_1 - \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\right), \tag{8.6}$$

$$\tau_{xz} = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right)\varphi, \qquad \tau_{yz} = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right)\psi, \tag{8.7}$$

where $\varphi(x, y)$ and $\psi(x, y)$ are undetermined functions of the coordinates x, y and Z_1 and Z_2 are determined from (2.29).

From the generalized Hooke's law (1.4.7) and using equations (8.7) in the expressions for transverse shear strains, we obtain:

$$e_{xz} = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right) a_{55}\varphi, \qquad e_{yz} = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right) a_{44}\psi. \tag{8.8}$$

On the basis of expressions (8.6) and (8.8) and using (8.1) and (8.2) we have for displacements:

$$u_z = u_z(x,y) = w(x,y), \tag{8.9}$$

$$\left.\begin{aligned} u_x &= u - z\frac{\partial w}{\partial x} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right) a_{55}\varphi, \\ u_y &= v - z\frac{\partial w}{\partial y} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right) a_{44}\psi, \end{aligned}\right\} \tag{8.10}$$

where w = w(x, y), u = u(x, y) and v = v(x, y) are the normal and tangential displacements of the plate middle plane.

Examination of equations (8.9) and (8.10) shows that the displacement formulas for u_z, u_x and u_y are identical to the corresponding formulas of the linear theory of bending of anisotropic plates, i.e., with formulas (2.12), (2.18) and (2.19).

Substituting values of displacements from (8.9) and (8.10) into the corresponding expressions (8.1) and (8.2), the following expressions are obtained for the deformations e_x, e_y and e_{xy}:

$$\left.\begin{aligned} e_x &= \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right) a_{55}\frac{\partial \varphi}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\ e_y &= \frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right) a_{44}\frac{\partial \psi}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \\ e_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{\partial x\,\partial y} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right) \times \\ &\quad \times \left(a_{55}\frac{\partial \varphi}{\partial y} + a_{44}\frac{\partial \psi}{\partial x}\right) + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}. \end{aligned}\right\} \tag{8.11}$$

Assuming that z = 0 and eliminating displacements u, v, w we obtain from (8.11) a known equation for the compatibility of the middle plane [37].

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} - \frac{\partial^2 \gamma}{\partial x\,\partial y} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}, \tag{8.12}$$

where

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \qquad \varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \\ \gamma &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}. \end{aligned}\right\} \tag{8.13}$$

From the generalized Hooke's law (1.4.7) and using (8.6) and (8.11) we obtain for the stresses

$$\sigma_x = B_{11}\left[\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + B_{12}\left[\frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(B_{11}a_{55}\frac{\partial \varphi}{\partial x} + B_{12}a_{44}\frac{\partial \psi}{\partial y}\right) - A_1\left[Z_1 - \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\right)\right], \tag{8.14}$$

$$\sigma_y = B_{22}\left[\frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{12}\left[\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(B_{22}a_{44}\frac{\partial \psi}{\partial y} + B_{12}a_{55}\frac{\partial \varphi}{\partial x}\right) - A_2\left[Z_1 - \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\right)\right], \tag{8.15}$$

$$\tau_{xy} = B_{66}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)B_{66}\left(a_{55}\frac{\partial \varphi}{\partial y} + a_{44}\frac{\partial \psi}{\partial x}\right). \tag{8.16}$$

In a similar manner the internal stress and moment resultants are obtained:

$$T_x = C_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + C_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] - hA_1Z_1, \tag{8.17}$$

$$T_y = C_{22}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + C_{12}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] - hA_2Z_1, \tag{8.18}$$

$$S = C_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right) \tag{8.19}$$

$$M_x = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} + \\ + \frac{h^2}{10}\left(D_{11}a_{55}\frac{\partial \varphi}{\partial x} + D_{12}a_{44}\frac{\partial \psi}{\partial y}\right) + A_1\frac{h^5}{120}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\right), \tag{8.20}$$

$$M_y = -D_{22}\frac{\partial^2 w}{\partial y^2} - D_{12}\frac{\partial^2 w}{\partial x^2} + \\ + \frac{h^2}{10}\left(D_{22}a_{44}\frac{\partial \psi}{\partial y} + D_{12}a_{55}\frac{\partial \varphi}{\partial x}\right) + A_2\frac{h^5}{120}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\right), \tag{8.21}$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial x\,\partial y} + \frac{h^2}{10}D_{66}\left(a_{55}\frac{\partial \varphi}{\partial y} + a_{44}\frac{\partial \psi}{\partial x}\right), \tag{8.22}$$

$$N_x = \frac{h^3}{12}\varphi, \qquad N_y = \frac{h^3}{12}\psi. \tag{8.23}$$

It should be remembered that

$$Z_1 = \frac{Z^+ - Z^-}{2}, \qquad Z_2 = Z^+ + Z^-, \tag{2.29}$$

$$C_{ik} = hB_{ik}, \qquad D_{ik} = \frac{h^3}{12}B_{ik}. \tag{2.44}$$

From (8.3) – (8.5) the following plate equilibrium equations with respect to internal stress and moment resultants are obtained:

$$\frac{\partial T_x}{\partial x} + \frac{\partial S}{\partial y} = 0, \qquad \frac{\partial S}{\partial x} + \frac{\partial T_y}{\partial y} = 0, \tag{8.24}$$

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + \frac{\partial}{\partial x}\left(T_x\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(S\frac{\partial w}{\partial x}\right) + \\ + \frac{\partial}{\partial x}\left(S\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(T_y\frac{\partial w}{\partial y}\right) = -Z_2, \tag{8.25}$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} = N_x, \qquad \frac{\partial M_y}{\partial y} + \frac{\partial H}{\partial x} = N_y. \tag{8.26}$$

If we assume that

$$T_x = \frac{\partial^2 F}{\partial y^2}, \qquad T_y = \frac{\partial^2 F}{\partial x^2}, \qquad S = -\frac{\partial^2 F}{\partial x\,\partial y}, \tag{8.27}$$

where F = F(x, y) is an undetermined stress function, then equations (8.24) are satisfied identically and then from (8.17) – (8.19) with consideration of (8.13) we obtain for ϵ_x, ϵ_y and γ:

$$\left.\begin{aligned}\varepsilon_x &= \frac{a_{11}}{h}\frac{\partial^2 F}{\partial y^2} + \frac{a_{12}}{h}\frac{\partial^2 F}{\partial x^2} + (a_{11}A_1 + a_{12}A_2)Z_1,\\ \varepsilon_y &= \frac{a_{22}}{h}\frac{\partial^2 F}{\partial x^2} + \frac{a_{12}}{h}\frac{\partial^2 F}{\partial y^2} + (a_{22}A_2 + a_{12}A_1)Z_1,\\ \gamma &= -\frac{a_{66}}{h}\frac{\partial^2 F}{\partial x\,\partial y},\end{aligned}\right\} \tag{8.28}$$

During the derivation of expressions (8.28) the identities (5.1), (5.2) and (2.44) were used.

Substituting values of the internal stress and moment resultants from (8.20) – (8.23) and (8.27) into equations (8.25) and (8.26), and of values of ϵ_x, ϵ_y and γ from (8.28) into the compatibility equation (8.12) we obtain

$$\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - 2\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2 F}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2} + \frac{h^3}{12}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\right) = -Z_2, \tag{8.29}$$

$$D_{11}\frac{\partial^3 w}{\partial x^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x\,\partial y^2} - \frac{h^2}{10}\left[a_{55}D_{11}\frac{\partial^2 \varphi}{\partial x^2} + a_{55}D_{66}\frac{\partial^2 \varphi}{\partial y^2} + \right.$$

$$\left. + a_{44}(D_{12}+D_{66})\frac{\partial^2 \psi}{\partial x\,\partial y}\right] + \frac{h^3}{12}\varphi - A_1\frac{h^5}{120}\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x\,\partial y}\right) = 0, \tag{8.30}$$

$$D_{22}\frac{\partial^3 w}{\partial y^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x^2\,\partial y} - \frac{h^2}{10}\left[a_{44}D_{22}\frac{\partial^2 \psi}{\partial y^2} + a_{44}D_{66}\frac{\partial^2 \psi}{\partial x^2} + \right.$$

$$\left. + a_{55}(D_{12}+D_{66})\frac{\partial^2 \varphi}{\partial x\,\partial y}\right] + \frac{h^3}{12}\psi - A_2\frac{h^5}{120}\left(\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial x\,\partial y}\right) = 0, \tag{8.31}$$

$$a_{22}\frac{1}{h}\frac{\partial^4 F}{\partial x^4} + (a_{66} + 2a_{12})\frac{1}{h}\frac{\partial^4 F}{\partial x^2\,\partial y^2} + a_{11}\frac{1}{h}\frac{\partial^4 F}{\partial y^4} -$$

$$-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 + \frac{\partial^2 w}{\partial x^2}\cdot\frac{\partial^2 w}{\partial y^2} = -\left[(a_{11}A_1 + a_{12}A_2)\frac{\partial^2}{\partial y^2} + \right.$$

$$\left. + (a_{22}A_2 + a_{12}A_1)\frac{\partial^2}{\partial x^2}\right]Z_1. \tag{8.32}$$

Thus, a complete system of four differential equations with respect to four undetermined functions w(x, y), φ(x, y), ψ(x, y) and F(x, y) is obtained.

The undetermined functions must satisfy the system of differential

equations (8.29) – (8.32) as well as the boundary conditions which, generally speaking, are of the familiar form (3.13) – (3.21). Boundary conditions will be defined more precisely in the next chapter when dealing with actual problems.

Finally we present the following expressions for the determination of inplane displacements of points of the middle plane obtained from (8.13) and (8.28):

$$\frac{\partial u}{\partial x} = -\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{a_{11}}{h}\frac{\partial^2 F}{\partial y^2} + \frac{a_{12}}{h}\frac{\partial^2 F}{\partial x^2} + (a_{11}A_1 + a_{12}A_2)Z_1, \quad (8.33)$$

$$\frac{\partial v}{\partial y} = -\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{a_{22}}{h}\frac{\partial^2 F}{\partial x^2} + \frac{a_{12}}{h}\frac{\partial^2 F}{\partial y^2} + (a_{22}A_2 + a_{12}A_1)Z_1. \quad (8.34)$$

When dealing with certain problems it is more convenient to use another system of differential equations in place of (8.29) – (8.32).

When the values of internal stress and moment resultants from (8.17) – (8.23) are substituted into (8.24) – (8.26), a complete system of five differential equations with respect to five undetermined functions u, v, w, φ, ψ is obtained:

$$C_{11}\frac{\partial^2 u}{\partial x^2} + (C_{12} + C_{66})\frac{\partial^2 v}{\partial x\,\partial y} + C_{66}\frac{\partial^2 u}{\partial y^2} + (C_{12} + C_{66})\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\,\partial y} + \\ + \frac{\partial w}{\partial x}\left(C_{11}\frac{\partial^2 w}{\partial x^2} + C_{66}\frac{\partial^2 w}{\partial y^2}\right) = hA_1\frac{\partial Z_1}{\partial x},$$

$$C_{22}\frac{\partial^2 v}{\partial y^2} + (C_{12} + C_{66})\frac{\partial^2 u}{\partial x\,\partial y} + C_{66}\frac{\partial^2 v}{\partial x^2} + \\ + (C_{12} + C_{66})\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial w}{\partial y}\left(C_{22}\frac{\partial^2 w}{\partial y^2} + C_{66}\frac{\partial^2 w}{\partial x^2}\right) = hA_2\frac{\partial Z_1}{\partial y},$$

$$\frac{h^3}{12}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\right) + \frac{3}{2}\left[C_{11}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + C_{22}\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2\right] + \\ + (C_{12} + 2C_{66})\left[\frac{1}{2}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial x\,\partial y} + \right. \\ \left. + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2\right] = -Z_2 + h\left[A_1\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}Z_1\right) + A_2\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}Z_1\right)\right].$$

The remaining two equations of this system are not given here because they are identical to equations (8.30) and (8.31).

The analogous equations for cylindrically orthotropic plates are given below (see Section 7):

For stresses

$$\sigma_r = B_r\left[\frac{\partial u}{\partial r} - z\frac{\partial^2 w}{\partial r^2} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2\right] +$$
$$+ B_r\nu_\vartheta\left[\frac{u}{r} + \frac{\partial v}{r\,\partial\vartheta} - z\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial\vartheta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \frac{1}{2r^2}\left(\frac{\partial w}{\partial\vartheta}\right)^2\right] +$$
$$+ \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left[B_r a_r\left(\frac{\partial\varphi}{\partial r} + \nu_\vartheta\frac{\varphi}{r}\right) + B_r\nu_\vartheta a_\vartheta\frac{\partial\psi}{r\,\partial\vartheta}\right] -$$
$$- A_1\left[Z_1 - \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(\frac{\partial\varphi}{\partial r} + \frac{\varphi}{r} + \frac{1}{r}\frac{\partial\psi}{\partial\vartheta}\right)\right], \qquad (8.35)$$

$$\sigma_\vartheta = B_\vartheta\left[\frac{\partial v}{r\,\partial\vartheta} + \frac{u}{r} - z\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial\vartheta^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + \frac{1}{2r^2}\left(\frac{\partial w}{\partial\vartheta}\right)^2\right] +$$
$$+ B_\vartheta\nu_r\left[\frac{\partial u}{\partial r} - z\frac{\partial^2 w}{\partial r^2} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2\right] +$$
$$+ \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left[B_\vartheta a_\vartheta\frac{\partial\psi}{r\,\partial\vartheta} + B_\vartheta a_r\left(\frac{\varphi}{r} + \nu_r\frac{\partial\varphi}{\partial r}\right)\right] -$$
$$- A_2\left[Z_1 - \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(\frac{\partial\varphi}{\partial r} + \frac{\varphi}{r} + \frac{1}{r}\frac{\partial\psi}{\partial\vartheta}\right)\right], \qquad (8.36)$$

$$\tau_{r\vartheta} = B_k\left\{\frac{\partial u}{r\,\partial\vartheta} + \frac{\partial v}{\partial r} - \frac{v}{r} - 2z\frac{\partial}{\partial r}\left(\frac{\partial w}{r\,\partial\vartheta}\right) + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial\vartheta}\right\} +$$
$$+ \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)B_k\left[a_r\frac{\partial\varphi}{r\,\partial\vartheta} + a_\vartheta\left(\frac{\partial\psi}{\partial r} - \frac{\psi}{r}\right)\right]. \qquad (8.37)$$

For internal stress and moment resultants:

$$T_r = C_r\left[\frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2\right] +$$
$$+ C_r\nu_\vartheta\left[\frac{1}{r}\frac{\partial v}{\partial\vartheta} + \frac{u}{r} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial\vartheta}\right)^2\right] - A_1 h Z_1, \qquad (8.38)$$

$$T_\vartheta = C_\vartheta\left[\frac{1}{r}\frac{\partial v}{\partial\vartheta} + \frac{u}{r} + \frac{1}{2r^2}\left(\frac{\partial w}{\partial\vartheta}\right)^2\right] +$$
$$+ C_\vartheta\nu_r\left[\frac{\partial u}{\partial r} + \frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2\right] - hA_2 Z_1. \qquad (8.39)$$

$$S_{\vartheta r} = C_k\left[\frac{1}{r}\frac{\partial u}{\partial\vartheta} + \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial\vartheta}\right]. \qquad (8.40)$$

$$M_r = -D_r\frac{\partial^2 w}{\partial r^2} - D_r\nu_\vartheta\frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\vartheta^2} + \frac{\partial w}{\partial r}\right) +$$

$$+\frac{h^2}{10}\left[D_r a_r\left(\frac{\partial\varphi}{\partial r}+\nu_\vartheta\frac{\varphi}{r}\right)+D_r\nu_\vartheta a_\vartheta\frac{1}{r}\frac{\partial\psi}{\partial\vartheta}\right]+$$

$$+\frac{h^5}{120}A_1\left(\frac{\partial\varphi}{\partial r}+\frac{\varphi}{r}+\frac{1}{r}\frac{\partial\psi}{\partial\vartheta}\right), \qquad (8.41)$$

$$M_\vartheta=-D_\vartheta\frac{1}{r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\vartheta^2}+\frac{\partial w}{\partial r}\right)-D_\vartheta\nu_r\frac{\partial^2 w}{\partial r^2}+$$

$$+\frac{h^2}{10}\left[D_\vartheta a_\vartheta\frac{1}{r}\frac{\partial\psi}{\partial\vartheta}+D_\vartheta a_r\left(\frac{\varphi}{r}+\nu_r\frac{\partial\varphi}{\partial r}\right)-\right.$$

$$\left.-\frac{h^5}{120}A_2\left(\frac{\partial\varphi}{\partial r}+\frac{\varphi}{r}+\frac{1}{r}\frac{\partial\psi}{\partial\vartheta}\right), \qquad (8.42)\right.$$

$$H_{r\vartheta}=D_k\left\{-2\frac{\partial}{\partial r}\left(\frac{\partial w}{r\,\partial\vartheta}\right)+\frac{h^2}{10}\left[a_r\frac{\partial\varphi}{r\,\partial\vartheta}+a_\vartheta\left(\frac{\partial\psi}{\partial r}-\frac{\psi}{r}\right)\right]\right\}, \qquad (8.43)$$

$$N_r=\frac{h^3}{12}\varphi, \qquad N_\vartheta=\frac{h^3}{12}\psi. \qquad (8.44)$$

The internal stress resultants T_r, T_θ, and $S_{\theta r}$ can be represented in the following manner in terms of the undetermined stress function $F = F(r, \theta)$:

$$T_r=\frac{1}{r^2}\frac{\partial^2 F}{\partial\vartheta^2}+\frac{1}{r}\frac{\partial F}{\partial r}, \quad T_\vartheta=\frac{\partial^2 F}{\partial r^2}, \quad S_{r\vartheta}=\frac{1}{r^2}\frac{\partial F}{\partial\vartheta}-\frac{1}{r}\frac{\partial^2 F}{\partial r\,\partial\vartheta}. \qquad (8.45)$$

Inplane displacements of points in the middle plane $u = u(r, \theta)$ and $v = v(r, \theta)$ can be determined from the following equations:

$$\left.\begin{aligned}\frac{\partial u}{\partial r}&=-\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2+\frac{a_{11}}{h}\left(\frac{1}{r^2}\frac{\partial^2 F}{\partial\vartheta^2}+\frac{1}{r}\frac{\partial F}{\partial r}\right)+\\&\quad+\frac{a_{12}}{h}\frac{\partial^2 F}{\partial r^2}+(A_1a_{11}+A_2a_{12})Z_1,\\\frac{1}{r}\frac{\partial v}{\partial\vartheta}+\frac{u}{r}&=-\frac{1}{2r^2}\left(\frac{\partial w}{\partial\vartheta}\right)^2+\frac{a_{12}}{h}\left(\frac{1}{r^2}\frac{\partial^2 F}{\partial\vartheta^2}+\frac{1}{r}\frac{\partial F}{\partial r}\right)+\\&\quad+\frac{a_{22}}{h}\frac{\partial^2 F}{\partial r^2}+(A_2a_{22}+A_1a_{12})Z_1.\end{aligned}\right\} \qquad (8.46)$$

Finally, the following system of four differential equations with respect to four undetermined functions is obtained:

$$\frac{h^3}{12}\left[\frac{\partial}{\partial r}(r\varphi)+\frac{\partial\psi}{\partial\vartheta}\right]+\frac{\partial}{\partial r}\left[\frac{\partial w}{\partial r}\left(\frac{1}{r}\frac{\partial^2 F}{\partial\vartheta^2}+\frac{\partial F}{\partial r}\right)\right]+$$
$$+\frac{\partial}{\partial\vartheta}\left[\frac{\partial w}{\partial r}\left(\frac{1}{r^2}\frac{\partial F}{\partial\vartheta}-\frac{1}{r}\frac{\partial^2 F}{\partial r\,\partial\vartheta}\right)\right]+$$
$$+\frac{\partial}{\partial r}\left[\frac{\partial w}{\partial\vartheta}\left(\frac{1}{r^2}\frac{\partial F}{\partial\vartheta}-\frac{1}{r}\frac{\partial^2 F}{\partial r\,\partial\vartheta}\right)\right]+\frac{\partial}{\partial\vartheta}\left(\frac{1}{r}\frac{\partial w}{\partial\vartheta}\frac{\partial^2 F}{\partial r^2}\right)=-rZ_2, \qquad (8.47)$$

$$D_r\frac{\partial}{\partial r}\left(r\frac{\partial^2 w}{\partial r^2}\right)+D_{r\vartheta}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial^2 w}{\partial\vartheta^2}\right)-D_\vartheta\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\vartheta^2}\right)-$$
$$-\frac{h^2}{10}\left\{a_r\left[D_r\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)+D_k\frac{1}{r}\frac{\partial^2\varphi}{\partial\vartheta^2}-D_\vartheta\frac{\varphi}{r}\right]+\right.$$
$$\left.+a_\vartheta\left[(D_\vartheta\nu_r+D_k)\frac{\partial^2\psi}{\partial r\,\partial\vartheta}-(D_\vartheta+D_k)\frac{1}{r}\frac{\partial\psi}{\partial\vartheta}\right]\right\}+$$
$$+\frac{h^3}{12}r\varphi-\frac{h^5}{120}\left[A_1\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}+\varphi+\frac{\partial\psi}{\partial\vartheta}\right)-\right.$$
$$\left.-A_2\left(\frac{\partial\varphi}{\partial r}+\frac{\varphi}{r}+\frac{1}{r}\frac{\partial\psi}{\partial\vartheta}\right)\right]=0, \qquad (8.48)$$

$$D_\vartheta\left(\frac{1}{r^2}\frac{\partial^3 w}{\partial\vartheta^3}+\frac{1}{r}\frac{\partial^2 w}{\partial r\,\partial\vartheta}\right)+D_{r\vartheta}\frac{\partial^3 w}{\partial r^2\,\partial\vartheta}-$$
$$-\frac{h^2}{10}\left\{a_\vartheta\left[D_\vartheta\frac{1}{r}\frac{\partial^2\psi}{\partial\vartheta^2}+D_k\left(r\frac{\partial^2\psi}{\partial r^2}+\frac{\partial\psi}{\partial r}-\frac{\psi}{r}\right)\right]+\right.$$
$$\left.+a_r\left[(D_r\nu_\vartheta+D_k)\frac{\partial^2\varphi}{\partial r\,\partial\vartheta}+(D_\vartheta+D_k)\frac{1}{r}\frac{\partial\varphi}{\partial\vartheta}\right]\right\}+\frac{h^3}{12}r\psi-$$
$$-\frac{h^5}{120}A_2\frac{\partial}{\partial\vartheta}\left(\frac{\partial\varphi}{\partial r}+\frac{\varphi}{r}+\frac{1}{r}\frac{\partial\psi}{\partial\vartheta}\right)=0, \qquad (8.49)$$

$$\frac{a_{22}}{h}\left(\frac{\partial^4 F}{\partial r^4}+\frac{2}{r}\frac{\partial^3 F}{\partial r^3}\right)+\left(\frac{a_{66}}{h}+2\frac{a_{12}}{h}\right)\times$$
$$\times\left(\frac{1}{r^2}\frac{\partial^4 F}{\partial r^2\,\partial\vartheta^2}-\frac{1}{r^3}\frac{\partial^3 F}{\partial r\,\partial\vartheta^2}+\frac{1}{r^4}\frac{\partial^2 F}{\partial\vartheta^2}\right)+$$
$$+\frac{a_{11}}{h}\left(\frac{1}{r^4}\frac{\partial^4 F}{\partial\vartheta^4}-\frac{1}{r^2}\frac{\partial^2 F}{\partial r^2}+\frac{1}{r^3}\frac{\partial F}{\partial r}+\frac{2}{r^4}\frac{\partial^2 F}{\partial\vartheta^2}\right)+$$
$$+\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial\vartheta^2}\right)-\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial\vartheta}\right)\right]^2+$$
$$+\left[(A_2a_{22}+A_1a_{12})\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)+\right.$$
$$\left.+(A_1a_{11}+A_2a_{12})\left(\frac{1}{r^2}\frac{\partial^2}{\partial\vartheta^2}-\frac{1}{r}\frac{\partial}{\partial r}\right)\right]Z_1=0. \qquad (8.50)$$

The boundary conditions are presented in the usual form, equations (3.19) – (3.21).

In the particular case when the effect of σ_z on the state of stress is not considered, the basic formulas and principal differential equations may be obtained from the corresponding formulas and equations by substituting $A_1 = 0$ and $A_2 = 0$. In this case the systems of differential equations (8.29) – (8.32) and (8.47) – (8.50) can be reduced to corresponding systems of two differential equations with respect to two undetermined functions. For example, using expressions (5.28) – (5.30) we satisfy identically equations (8.30), (8.31) and from (8.29), (8.32) the desired system of two differential equations with respect to the two undetermined functions F(x, y) and Φ(x, y) is obtained. However, this system will be of ninth order, as opposed to the corresponding eighth order system of the classical theory.

2. Particular theory [38, 39]. This theory is based on the following assumptions:

   (a) the normal displacement u_z with respect to the plate middle plane does not depend on the z coordinate;

   (b) in dealing with deformation it is considered that the shear stresses τ_{xz} and τ_{yz} do not differ from the corresponding stresses determined from the classical theory of bending of orthotropic plates;

   (c) the effect of normal stresses σ_z is not considered during the determination of deformations because the stresses σ_z are considered to be negligibly small in comparison with the inplane stresses.

If it is desired to account for the effect of σ_z on the deformed and stressed state of the plate, then the terms with multipliers A_1 and A_2 from (8.14) – (8.50) should be added to the corresponding formulas and equations.

By accepting assumptions (a), (b) and (c) we assume in fact that

$$\frac{\partial u_z}{\partial z} = 0, \tag{8.51}$$

$$\left.\begin{aligned} \tau_{xz} &= \frac{1}{2}\left(\frac{h^2}{4} - z^2\right)\varphi_0, \quad \tau_{yz} = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right)\psi_0, \\ \varphi_0 &= -\left[B_{11}\frac{\partial^3 w_0}{\partial x^3} + (B_{12} + 2B_{66})\frac{\partial^3 w_0}{\partial x\,\partial y^2}\right], \\ \psi_0 &= -\left[B_{22}\frac{\partial^3 w_0}{\partial y^3} + (B_{12} + 2B_{66})\frac{\partial^3 w_0}{\partial y\,\partial x^2}\right], \end{aligned}\right\} \tag{8.52}$$

where w_0 is the normal displacement of the corresponding plate as determined from the classical theory, i.e., from the following system of equations

$$D_{11}\frac{\partial^4 w_0}{\partial x^4}+2(D_{12}+2D_{66})\frac{\partial^4 w_0}{\partial x^2\,\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}-$$
$$-\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 F_0}{\partial y^2}+2\frac{\partial^2 w_0}{\partial x\,\partial y}\frac{\partial^2 F_0}{\partial x\,\partial y}-\frac{\partial^2 w_0}{\partial y^2}\frac{\partial^2 F_0}{\partial x^2}=Z_2,$$

$$a_{22}\frac{1}{h}\frac{\partial^4 F_0}{\partial x^4}+(a_{66}+2a_{12})\frac{1}{h}\frac{\partial^4 F_0}{\partial x^2\,\partial y^2}+a_{11}\frac{1}{h}\frac{\partial^4 F_0}{\partial y^4}-$$
$$-\left(\frac{\partial^2 w_0}{\partial x\,\partial y}\right)^2+\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}=0,$$

and the corresponding boundary conditions.

In this system of equations $F_0(x, y)$ represents the stress function which expresses the internal stress resultants of the classical theory in the following manner:

$$T_x^0=\frac{\partial^2 F_0}{\partial y^2},\qquad T_y^0=\frac{\partial^2 F_0}{\partial x^2},\qquad S^0=-\frac{\partial^2 F_0}{\partial x\,\partial y}.$$

First we determine the displacements of any point of the plate:

$$u_z=w(x, y), \tag{8.53}$$

$$u_x=-z\frac{\partial w}{\partial x}+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)(a_{55}\varphi_0+a_{45}\psi_0), \tag{8.54}$$

$$u_y=-z\frac{\partial w}{\partial y}+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)(a_{44}\psi_0+a_{45}\varphi_0), \tag{8.55}$$

where $w = w(x, y)$ is the undetermined normal displacement of the plate middle plane.

For stresses we have

$$\sigma_x=B_{11}\left[\frac{\partial u}{\partial x}-z\frac{\partial^2 w}{\partial x^2}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right]+$$
$$+B_{12}\left[\frac{\partial v}{\partial y}-z\frac{\partial^2 w}{\partial y^2}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right]+$$
$$+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\left(B_{11}a_{55}\frac{\partial\varphi_0}{\partial x}+B_{12}a_{44}\frac{\partial\psi_0}{\partial y}\right). \tag{8.56}$$

$$\sigma_y = B_{22}\left[\frac{\partial v}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + B_{12}\left[\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(B_{22}a_{44}\frac{\partial \psi_0}{\partial y} + B_{12}a_{55}\frac{\partial \varphi_0}{\partial x}\right), \tag{8.57}$$

$$\tau_{xy} = B_{66}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right] + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)B_{66}\left(a_{55}\frac{\partial \varphi_0}{\partial y} + a_{44}\frac{\partial \psi_0}{\partial x}\right). \tag{8.58}$$

The internal stress resultants and moments are:

$$\left.\begin{aligned} T_x &= C_{11}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] + C_{12}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right], \\ T_y &= C_{22}\left[\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] + C_{12}\left[\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right], \\ S &= C_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right); \end{aligned}\right\} \tag{8.59}$$

$$\left.\begin{aligned} M_x &= -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} + \frac{h^2}{10}\left(D_{11}a_{55}\frac{\partial \varphi_0}{\partial x} + D_{12}a_{44}\frac{\partial \psi_0}{\partial y}\right), \\ M_y &= -D_{22}\frac{\partial^2 w}{\partial y^2} - D_{12}\frac{\partial^2 w}{\partial x^2} + \frac{h^2}{10}\left(D_{22}a_{44}\frac{\partial \psi_0}{\partial y} + D_{12}a_{55}\frac{\partial \varphi_0}{\partial x}\right), \\ H &= -2D_{66}\frac{\partial^2 w}{\partial x\,\partial y} + \frac{h^2}{10}D_{66}\left(a_{55}\frac{\partial \varphi_0}{\partial y} + a_{44}\frac{\partial \psi_0}{\partial x}\right). \end{aligned}\right\} \tag{8.60}$$

and from equation (8.26) we obtain the transverse shear resultants:

$$\left.\begin{aligned} N_x &= -\left[D_{11}\frac{\partial^3 w}{\partial x^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x\,\partial y^2}\right] + \\ &+ \frac{h^2}{10}a_{55}\left(D_{11}\frac{\partial^2}{\partial x^2} + D_{66}\frac{\partial^2}{\partial y^2}\right)\varphi_0 + \frac{h^2}{10}a_{44}(D_{12} + D_{66})\frac{\partial^2 \psi_0}{\partial x\,\partial y}, \\ N_y &= -\left[D_{22}\frac{\partial^3 w}{\partial y^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial y\,\partial x^2}\right] + \\ &+ \frac{h^2}{10}a_{44}\left(D_{22}\frac{\partial^2}{\partial y^2} + D_{66}\frac{\partial^2}{\partial x^2}\right)\psi_0 + \frac{h^2}{10}a_{55}(D_{12} + D_{66})\frac{\partial^2 \varphi_0}{\partial x\,\partial y}. \end{aligned}\right\} \tag{8.61}$$

From the formulas for internal forces (8.59) – (8.61) it is evident that the correction terms for the classical theory involve only the moment and transverse shear resultants, while the non-linear terms are present in only the inplane stress resultants T_x, T_y and S, and these non-linear terms have the same form as the classical theory.

On the assumption that

$$T_x = \frac{\partial^2 F}{\partial y^2}, \quad T_y = \frac{\partial^2 F}{\partial x^2}, \quad S = -\frac{\partial^2 F}{\partial x\, \partial y}, \tag{8.62}$$

where, as before, F = F(x, y) is the undetermined stress function which satisfies the equilibrium equations (8.24), we obtain from (8.59) after considering (8.13):

$$\left.\begin{aligned} \varepsilon_x &= \frac{a_{11}}{h}\frac{\partial^2 F}{\partial y^2} + \frac{a_{12}}{h}\frac{\partial^2 F}{\partial x^2}, \\ \varepsilon_y &= \frac{a_{22}}{h}\frac{\partial^2 F}{\partial x^2} + \frac{a_{12}}{h}\frac{\partial^2 F}{\partial y^2}, \end{aligned} \qquad \gamma = -\frac{a_{66}}{h}\frac{\partial^2 F}{\partial x\, \partial y}. \right\} \tag{8.63}$$

From (8.25) with the use of (8.26) to eliminate transverse stress resultants N_x and N_y we obtain:

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 H}{\partial x\, \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial}{\partial x}\left(T_x \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial y}\left(S \frac{\partial w}{\partial x}\right) + \\ + \frac{\partial}{\partial x}\left(S \frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(T_y \frac{\partial w}{\partial y}\right) = -Z_2. \tag{8.64}$$

When values of ϵ_x, ϵ_y and γ from (8.63) are substituted into (8.12) and the values for the moment and inplane stress resultants from (8.60) and (8.62) are substituted into the equilibrium equation (8.64), a system of two differential equations with respect to the two undetermined functions w = w(x, y) and F = F(x, y) is obtained:

$$a_{22}\frac{1}{h}\frac{\partial^4 F}{\partial x^4} + (a_{66} + 2a_{12})\frac{1}{h}\frac{\partial^4 F}{\partial x^2\, \partial y^2} + a_{11}\frac{1}{h}\frac{\partial^4 F}{\partial y^4} - \\ - \left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} = 0, \tag{8.65}$$

$$D_{11}\frac{\partial^4 w}{\partial x^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x^2\, \partial y^2} + D_{22}\frac{\partial^4 w}{\partial y^4} -$$

$$-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2}+2\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2 F}{\partial x\,\partial y}-\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2}=$$

$$=Z_2+\frac{h^2}{10}a_{55}\left[D_{11}\frac{\partial^3}{\partial x^3}+(D_{12}+2D_{66})\frac{\partial^3}{\partial x\,\partial y^2}\right]\varphi_0+$$

$$+\frac{h^2}{10}a_{44}\left[D_{22}\frac{\partial^3}{\partial y^3}+(D_{12}+2D_{66})\frac{\partial^3}{\partial y\,\partial x^2}\right]\psi_0. \qquad (8.66)$$

It is evident upon examining (8.65) and (8.66) that equation (8.65) is completely identical to the corresponding equation of the classical theory, and the left hand side of equation (8.66) is identical to the corresponding left hand side of the equations of the classical theory. The right hand side of equation (8.66) contains a correction to Z_2 which can be easily derived, provided the solution to the classical problem under consideration is known.

It should be remembered that the boundary conditions in this particular theory, generally speaking, are the same as the boundary conditions of the classical theory [40].

The equations of the non-linear particular theory for the case of cylindrical anisotropy are not given here because they can be easily derived by analogy. It is necessary only to remember equations (7.30) – (7.35) and (8.35) – (8.50).

9. Theory for the Analysis of Two-Layer Orthotropic Plates [41, 42]

Consider a two-layer orthotropic plate made of two uniform orthotropic materials. The planes of elastic symmetry of each layer are mutually perpendicular. One of these planes at each point of every layer is parallel to the external parallel planes of the plate, and the remaining two are perpendicular to the coordinate lines x = constant, y = constant.

We assume that the coordinate plane Oxy coincides with the intersection plane of the layers and is thus also parallel to the external planes of the plate (Figure 10.).

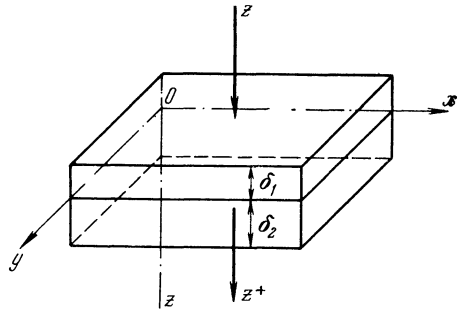

Figure 10.

It is also considered that the layers displace together along the intersecting plane and remain elastic.

The following assumptions are used in this theory of two-layer plates:

(a) a displacement u_z in each plate layer which is normal to the plate middle plane does not depend on the z coordinate;

(b) in the determination of deformation it is considered that shearing stresses τ_{xz}^{m}, τ_{yz}^{m} are the same as the corresponding stresses ($\tau_{xz}^{m,\,o}$, $\tau_{yz}^{m,\,o}$) found with the assumption of nondeformable normals for the entire plate as a whole;

(c) the effect of normal stresses σ_z is not considered during the determination of deformation because σ_z is negligibly small in comparison with the inplane stresses.

Similar assumptions are found in many works [43–47]. Using assumptions (a), (b) and (c) we assume in fact that

$$e_z^{m} = 0, \tag{9.1}$$

$$\tau_{xz}^{m} = \tau_{xz}^{m,\,0} = -\frac{1}{2}(\delta_m^2 - z^2)\left[B_{11}^{m}\frac{\partial^3 w_0}{\partial x^3} + (B_{12}^{m} + 2B_{66}^{m})\frac{\partial^3 w_0}{\partial x \partial y^2}\right] +$$
$$+ [(-1)^m \delta_m - z]\left[B_{11}^{m}\frac{\partial^2 u_0}{\partial x^2} + B_{66}^{m}\frac{\partial^2 u_0}{\partial y^2} + (B_{12}^{m} + B_{66}^{m})\frac{\partial^2 v_0}{\partial x\,\partial y}\right], \tag{9.2}$$

$$\tau_{yz}^{m} = \tau_{yz}^{m,\,0} = -\frac{1}{2}(\delta_m^2 - z^2)\left[B_{22}^{m}\frac{\partial^3 w_0}{\partial y^3} + (B_{12}^{m} + 2B_{66}^{m})\frac{\partial^3 w_0}{\partial y\,\partial x^2}\right] +$$
$$+ [(-1)^m \delta_m - z]\left[B_{22}^{m}\frac{\partial^2 v_0}{\partial y^2} + B_{66}^{m}\frac{\partial^2 v_0}{\partial x^2} + (B_{12}^{m} + B_{66}^{m})\frac{\partial^2 u_0}{\partial x\,\partial y}\right], \tag{9.3}$$

where $u_0 = u_0(x, y)$, $v_0 = v_0(x, y)$, $w_0 = w_0(x, y)$ are tangential and normal displacements of the intersection plane in the case of classical theory, respectively; m is the number indicating the layer number. It should be noted that for the first and second layers when m is used as a superscript on the value under consideration, we use one or two prime signs instead of m = 1 or m = 2. Furthermore, in the expressions $(-1)^m$, and only in those expressions, m is considered as a power.

Examination of (9.2) and (9.3) shows that the determination of tangential stresses is reduced to the determination of displacements u_0, v_0 and w_0. These displacements are determined from the system of three differential equations [48, 49]:

$$\left.\begin{aligned} L_{11}(C_{ik})u_0 + L_{12}(C_{ik})v_0 - L_{13}(K_{ik})w_0 &= 0,\\ L_{22}(C_{ik})v_0 + L_{12}(C_{ik})u_0 - L_{23}(K_{ik})w_0 &= 0,\\ L_{33}(D_{ik})w_0 - L_{13}(K_{ik})u_0 - L_{23}(K_{ik})v_0 &= Z_2, \end{aligned}\right\} \tag{9.4}$$

where the linear operators are defined to be:

$$L_{11}(C_{ik}) = C_{11}\frac{\partial^2}{\partial x^2} + C_{66}\frac{\partial^2}{\partial y^2}, \tag{9.5}$$

$$L_{22}(C_{ik}) = C_{22}\frac{\partial^2}{\partial y^2} + C_{66}\frac{\partial^2}{\partial x^2}, \tag{9.6}$$

$$L_{12}(C_{ik}) = (C_{12} + C_{66})\frac{\partial^2}{\partial x\,\partial y}, \tag{9.7}$$

$$L_{13}(K_{ik}) = K_{11}\frac{\partial^3}{\partial x^3} + (K_{12} + 2K_{66})\frac{\partial^3}{\partial x\,\partial y^2}, \tag{9.8}$$

$$L_{23}(K_{ik}) = K_{22}\frac{\partial^3}{\partial y^3} + (K_{12} + 2K_{66})\frac{\partial^3}{\partial y\,\partial x^2}, \tag{9.9}$$

$$L_{33}(D_{ik}) = D_{11}\frac{\partial^4}{\partial x^4} + D_{22}\frac{\partial^4}{\partial y^4} + 2(D_{12} + 2D_{66})\frac{\partial^4}{\partial x^2\,\partial y^2}. \tag{9.10}$$

The following notation is used for the stiffness terms:

$$C_{ik} = (\delta_2 B_{ik}'' + \delta_1 B_{ik}'), \tag{9.11}$$

$$K_{ik} = \frac{1}{2}(\delta_2^2 B_{ik}'' - \delta_1^2 B_{ik}'), \tag{9.12}$$

$$D_{ik} = \frac{1}{3}(\delta_2^3 B_{ik}'' + \delta_1^3 B_{ik}'). \tag{9.13}$$

For the displacements of any point of the m-th layer of the plate (see Sections 2 and 6) we have:

$$u_z = w(x, y), \tag{9.14}$$

$$\left.\begin{aligned} u_x &= u - z\frac{\partial w}{\partial x} + z\varphi_1^{m,0} + z^2\zeta_1^{m,0} - \frac{z^3}{6}\chi_1^{m,0},\\ u_y &= v - z\frac{\partial w}{\partial y} + z\varphi_2^{m,0} + z^2\zeta_2^{m,0} - \frac{z^3}{6}\chi_2^{m,0}, \end{aligned}\right\} \tag{9.15}$$

where u = u(x, y), v = v(x, y), w = w(x, y) are the undetermined tangential and normal displacements of the intersection plane of the plate and:

$$\varphi_i^{m,0} = (-1)^m \delta_m \zeta_i^{m,0} + \frac{\delta_m^2}{2} \chi_i^{m,0}, \tag{9.16}$$

$$\left.\begin{aligned} \zeta_1^{m,0} &= a_{55}^m [L_{11}(B_{ik}^m) u_0 + L_{12}(B_{ik}^m) v_0], \\ \zeta_2^{m,0} &= a_{44}^m [L_{22}(B_{ik}^m) v_0 + L_{12}(B_{ik}^m) u_0], \end{aligned}\right\} \tag{9.17}$$

$$\chi_1^{m,0} = -a_{55}^m L_{13}(B_{ik}^m) w_0; \quad \chi_2^{m,0} = -a_{44}^m L_{23}(B_{ik}^m) w_0. \tag{9.18}$$

It is evident from equations (9.14) and (9.15) that in the case of a one-layer plate a normal displacement of any point of the m-th layer does not depend on the z coordinate, while the tangential displacements u_x and u_y depend non-linearly on z. The coefficients of the non-linear terms are known values which are determined from (9.17) and (9.18) on the basis of the theory of laminated plates, using the hypothesis of nondeformable normals for the entire plate as a whole.

Examination of (9.1) – (9.3), (9.4), (9.14) – (9.18) shows that the shearing stresses τ_{xz}^m, τ_{yz}^m, normal stress σ_z, and the displacements u_x, u_y, u_z satisfy boundary conditions on the surface and also the boundary conditions for the intersection of the two plate layers which are of the following form:

on external planes

$$\left.\begin{aligned} &\text{at } z = -\delta_1 \quad \tau_{xz}' = 0, \quad \tau_{yz}' = 0, \quad \sigma_z' = -Z^-, \\ &\text{at } z = \delta_2 \quad\;\; \tau_{xz}'' = 0, \quad \tau_{yz}'' = 0, \quad \sigma_z'' = Z^+; \end{aligned}\right\} \tag{9.19}$$

on the intersection plane at z = 0

$$\left.\begin{aligned} \tau_{xz}' &= \tau_{xz}'', \quad \tau_{yz}' = \tau_{yz}'', \quad \sigma_z' = \sigma_z'', \\ u_x' &= u_x'', \quad u_y' = u_y'', \quad u_z' = u_z''. \end{aligned}\right\} \tag{9.20}$$

Substituting the values of displacements u_x, u_y, and u_z from (9.14), (9.15) into (1.2.2) we obtain expressions for the principal deformations of the m-th layer

$$\left.\begin{aligned} e_x^m &= \varepsilon_1^m + z\varkappa_1^m + z^2\eta_1^m + z^3\vartheta_1^m, \\ e_y^m &= \varepsilon_2^m + z\varkappa_2^m + z^2\eta_2^m + z^3\vartheta_2^m, \\ e_{xy}^m &= \omega^m + z\tau^m + z^2\xi^m + z^3\lambda^m; \end{aligned}\right\} \tag{9.21}$$

where

$$\varepsilon_1^m = \varepsilon_1 = \frac{\partial u}{\partial x}, \quad \varepsilon_2^m = \varepsilon_2 = \frac{\partial v}{\partial y}, \quad \omega^m = \omega = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}; \tag{9.22}$$

$$\left.\begin{aligned} \varkappa_1^m &= -\frac{\partial^2 w}{\partial x^2} + \frac{\partial \varphi_1^{m,0}}{\partial x}, \quad \varkappa_2^m = -\frac{\partial^2 w}{\partial y^2} + \frac{\partial \varphi_2^{m,0}}{\partial y}, \\ \tau^m &= -2\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial \varphi_1^{m,0}}{\partial y} + \frac{\partial \varphi_2^{m,0}}{\partial x}; \end{aligned}\right\} \tag{9.23}$$

$$\left.\begin{aligned} \eta_1^m &= -\frac{1}{2}\frac{\partial \zeta_1^{m,0}}{\partial x}, \quad \eta_2^m = -\frac{1}{2}\frac{\partial \zeta_2^{m,0}}{\partial y}, \\ \xi^m &= -\frac{1}{2}\left(\frac{\partial \zeta_1^{m,0}}{\partial y} + \frac{\partial \zeta_2^{m,0}}{\partial x}\right); \end{aligned}\right\} \tag{9.24}$$

$$\left.\begin{aligned} \vartheta_1^m &= -\frac{1}{6}\frac{\partial \chi_1^{m,0}}{\partial x}, \quad \vartheta_2^m = -\frac{1}{6}\frac{\partial \chi_2^{m,0}}{\partial y}, \\ \lambda^m &= -\frac{1}{6}\left(\frac{\partial \chi_1^{m,0}}{\partial y} + \frac{\partial \chi_2^{m,0}}{\partial x}\right). \end{aligned}\right\} \tag{9.25}$$

Using equations (9.21) – (9.25) and the generalized Hooke's law we obtain for stresses:

$$\begin{aligned} \sigma_x^m = {} & (B_{11}^m \varepsilon_1 + B_{12}^m \varepsilon_2) + z(B_{11}^m \varkappa_1 + B_{12}^m \varkappa_2) + \\ & + B_{11}^m \frac{\partial}{\partial x}\left(z\varphi_1^{m,0} - \frac{z^2}{2}\zeta_1^{m,0} - \frac{z^3}{6}\chi_1^{m,0}\right) + \\ & + B_{12}\frac{\partial}{\partial y}\left(z\varphi_2^{m,0} - \frac{z^2}{2}\zeta_2^{m,0} - \frac{z^3}{6}\chi_2^{m,0}\right), \end{aligned} \tag{9.26}$$

$$\begin{aligned} \sigma_y^m = {} & (B_{22}^m \varepsilon_2 + B_{12}^m \varepsilon_1) + z(B_{22}^m \varkappa_2 + B_{12}^m \varkappa_1) + \\ & + B_{22}^m \frac{\partial}{\partial y}\left(z\varphi_2^{m,0} - \frac{z^2}{2}\zeta_2^{m,0} - \frac{z^3}{6}\chi_2^{m,0}\right) + \\ & + B_{12}^m \frac{\partial}{\partial x}\left(z\varphi_1^{m,0} - \frac{z^2}{2}\zeta_1^{m,0} - \frac{z^3}{6}\chi_1^{m,0}\right), \end{aligned} \tag{9.27}$$

$$\begin{aligned} \tau_{xy}^m = {} & B_{66}^m \omega + zB_{66}^m \tau + B_{66}^m \frac{\partial}{\partial y}\left(z\varphi_1^{m,0} - \frac{z^2}{2}\zeta_1^{m,0} - \frac{z^3}{6}\chi_1^{m,0}\right) + \\ & + B_{66}^m \frac{\partial}{\partial x}\left(z\varphi_2^{m,0} - \frac{z^2}{2}\zeta_2^{m,0} - \frac{z^3}{6}\chi_2^{m,0}\right), \end{aligned} \tag{9.28}$$

where along with expressions (9.22) – (9.25) we have

$$\varkappa_1 = -\frac{\partial^2 w}{\partial x^2} \qquad \varkappa_2 = -\frac{\partial^2 w}{\partial y^2}, \qquad \tau = -2\frac{\partial^2 w}{\partial x\,\partial y}. \tag{9.29}$$

For internal stress and moment resultants:

$$\left.\begin{aligned} T_x &= \int_{-\delta_1}^{0} \sigma_x' dz + \int_{0}^{\delta_2} \sigma_x'' dz, \\ H &= \int_{-\delta_1}^{0} \tau_{xy}' z\, dz + \int_{0}^{\delta_2} \tau_{xy}'' z\, dz. \end{aligned}\right\} \tag{9.30}$$

Substituting the values of stresses from (9.26) – (9.28) into (9.30) for internal stress and moment resultants we obtain:

$$T_x = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + K_{11}\varkappa_1 + K_{12}\varkappa_2 + T_x^*, \tag{9.31}$$

$$T_y = C_{22}\varepsilon_2 + C_{12}\varepsilon_1 + K_{22}\varkappa_2 + K_{12}\varkappa_1 + T_y^*, \tag{9.32}$$

$$S = C_{66}\omega + K_{66}\tau + S^*, \tag{9.33}$$

$$M_x = D_{11}\varkappa_1 + D_{12}\varkappa_2 + K_{11}\varepsilon_1 + K_{12}\varepsilon_2 + M_x^*, \tag{9.34}$$

$$M_y = D_{22}\varkappa_2 + D_{12}\varkappa_1 + K_{22}\varepsilon_2 + K_{12}\varepsilon_1 + M_y^*, \tag{9.35}$$

$$H = D_{66}\tau + K_{66}\omega + H^*, \tag{9.36}$$

where the correction terms (with stars) which are derived for the classical theory are:

$$\left.\begin{aligned} T_x^* &= \frac{\partial}{\partial x}(B_{11}'' R_1'' - B_{11}' R_1') + \frac{\partial}{\partial y}(B_{12}'' R_2'' - B_{12}' R_2'), \\ T_y^* &= \frac{\partial}{\partial y}(B_{22}'' R_2'' - B_{22}' R_2') + \frac{\partial}{\partial x}(B_{12}'' R_1'' - B_{12}' R_1'), \\ S^* &= B_{66}''\left(\frac{\partial R_1''}{\partial y} + \frac{\partial R_2''}{\partial x}\right) - B_{66}'\left(\frac{\partial R_1'}{\partial y} + \frac{\partial R_2'}{\partial x}\right); \end{aligned}\right\} \tag{9.37}$$

$$\left.\begin{aligned} M_x^* &= \frac{\partial}{\partial x}(B_{11}'' P_1'' + B_{11}' P_1') + \frac{\partial}{\partial y}(B_{12}'' P_2'' + B_{12}' P_2'), \\ M_y^* &= \frac{\partial}{\partial y}(B_{22}'' P_2'' + B_{22}' P_2') + \frac{\partial}{\partial x}(B_{12}'' P_1'' + B_{12}' P_1'), \\ H^* &= B_{66}\left(\frac{\partial P_1''}{\partial y} + \frac{\partial P_2''}{\partial x}\right) + B_{66}'\left(\frac{\partial P_1'}{\partial y} + \frac{\partial P_2'}{\partial x}\right); \end{aligned}\right\} \tag{9.38}$$

$$R_i^m = \frac{\delta_m^2}{2}\varphi_i^{m,0} - (-1)^m \frac{\delta_m^3}{6}\zeta_i^{m,0} - \frac{\delta_m^4}{24}\chi_i^{m,0}, \tag{9.39}$$

$$P_i^m = \frac{\delta_m^3}{3}\varphi_i^{m,0} - (-1)^m \frac{\delta_m^4}{8}\zeta_i^{m,0} - \frac{\delta_m^5}{30}\chi_i^{m,0}. \tag{9.40}$$

The internal stress and moment resultants should satisfy the equilibrium equations. The equilibrium equations for the laminated plate resemble those of (2.45) – (2.49). Eliminating the transverse shear resultants N_x and N_y from the system of five equations, the following three equilibrium equations for the differential element of the laminated plate are obtained:

$$\left.\begin{aligned} &\frac{\partial T_x}{\partial x} + \frac{\partial S}{\partial y} = 0, \quad \frac{\partial T_y}{\partial y} + \frac{\partial S}{\partial x} = 0, \\ &\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 H}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} = -Z_2. \end{aligned}\right\} \tag{9.41}$$

Substituting the values of internal stress and moment resultants from (9.31) – (9.36) into (9.41), a complete system of three differential equations with respect to three undetermined functions u(x, y), v(x, y) and w(x, y) is obtained:

$$L_{11}(C_{ik})u + L_{12}(C_{ik})v - L_{13}(K_{ik})w = -\frac{\partial T_x^*}{\partial x} - \frac{\partial S^*}{\partial y}, \tag{9.42}$$

$$L_{22}(C_{ik})v + L_{12}(C_{ik})u - L_{23}(K_{ik})w = -\frac{\partial T_y^*}{\partial y} - \frac{\partial S^*}{\partial x}, \tag{9.43}$$

$$L_{33}(D_{ik})w - L_{13}(K_{ik})u - L_{23}(K_{ik})v = Z_2 + \frac{\partial^2 M_x^*}{\partial x^2} + 2\frac{\partial^2 H^*}{\partial x\,\partial y} + \frac{\partial^2 M_y^*}{\partial y^2}. \tag{9.44}$$

The left hand sides of equations (9.42) – (9.44) are of the same form as the corresponding equations of the classical theory (9.4). Additional terms which define more precisely the actual solution than the classical theory, are present in the form of given load terms in the right hand sides of (9.42) – (9.44). (The additional terms are derived on the basis of classical theory).

The boundary conditions here coincide with the boundary conditions of the classical theory of laminated plates, i.e., with conditions (6.17) – (6.21) [50]. However, for the case of the theory suggested here these conditions can be expanded.

Finally, the theory suggested here for two-layer plates can be easily generalized for three-layer plates [51].

10. Theory for the Analysis of Multi-Layer Orthotropic Plates

Consider a thin multi-layer plate consisting of an odd number (2m + 1) of

uniform orthotropic layers arranged symmetrically with respect to the plate middle plane Oxy. These layers are of identical physical and mechanical properties (Figure 11.).

Let the plate be fabricated in such a way that one of the planes of elastic symmetry is parallel to the middle plane at each point of every layer, and the remaining two are perpendicular to the coordinate lines x = constant and y = constant.

It is assumed that all layers remain elastic and do not slip during deformation.

Furthermore, the plate is loaded only with normally applied surface load $Z^+ = Z^+(x, y)$, $Z^- = Z^-(x, y)$ (Figure 4.).

The theory of multi-layer plates is based on the following assumptions:

(a) the displacement u_z normal to the middle plane in each plate layer is independent of the z coordinate;

(b) shearing stresses τ_{xz}^i and τ_{yz}^i or corresponding deformations e_{xz}^i, e_{yz}^i change according to a given law with respect to the thickness of each layer.

(c) the effect of normal stresses σ_z^i is disregarded during the determination of the displacements u_x^i and u_y^i as well as in determining stresses σ_x^i and σ_y^i.

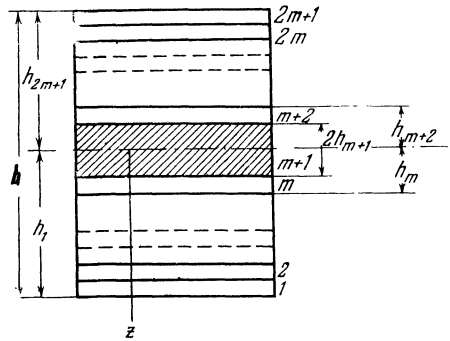

Figure 11.

On the basis of assumptions (a) and (b), for the i-th layer it is assumed that approximately

$$e_z^i = \frac{\partial u_z^i}{\partial z} = 0, \quad u_z^i = u_z^i(x, y) = w(x, y), \tag{10.1}$$

$$\tau_{xz}^i = [G_{13}^i f(z) + A_i]\varphi(x, y), \quad \tau_{yz}^i = [G_{23}^i f(z) + B_i]\psi(x, y), \tag{10.2}$$

where $\varphi(x, y)$, $\psi(x, y)$ are arbitrary undetermined functions of the x and y coordinates; f(z) is the function which characterizes changes of the shearing stresses τ_{xz} and τ_{yz} with respect to the plate thickness; at the same time f(z) =

f(−z), f(h/2) = 0; A_i, B_i are constants which are determined from the boundary conditions of adjacent layers.

The boundary conditions on the external planes, at the intersection of adjacent layers, and at the middle plane are represented as follows:

$$\left.\begin{aligned} \tau_{xz}^{1}\left(\frac{h}{2}\right) = \tau_{xz}^{2m+1}\left(-\frac{h}{2}\right) = 0, \quad \tau_{xz}^{i}(h_{i-1}) = \tau_{xz}^{i-1}(h_{i-1}), \\ \tau_{yz}^{1}\left(\frac{h}{2}\right) = \tau_{yz}^{2m+1}\left(-\frac{h}{2}\right) = 0, \quad \tau_{yz}^{i}(h_{i-1}) = \tau_{yz}^{i-1}(h_{i-1}), \end{aligned}\right\} \tag{10.3}$$

$$\left.\begin{aligned} u_x^i(h_{i-1}) = u_x^{i-1}(h_{i-1}), \\ u_x(x, y, z = 0) = u = 0, \\ u_y^i(h_{i-1}) = u_y^{i-1}(h_{i-1}), \\ u_y(x, y, z = 0) = v = 0. \end{aligned}\right\} \tag{10.4}$$

When the boundary conditions at the intersection of adjacent layers and those on the external planes (10.3) are satisfied, the following formulas are obtained for the constants A_i and B_i:

$$\left.\begin{aligned} A_1 = A_{2m+1} = 0, \quad B_1 = B_{2m+1} = 0, \\ A_i = \sum_{r=i}^{2m} f(h_r)\,(G_{13}^{(r+1)} - G_{13}^{(r)}), \quad i = m+1, \ldots, 2m, \\ B_i = \sum_{r=i}^{2m} f(h_r)\,(G_{23}^{(r+1)} - G_{23}^{(r)}), \quad i = m+1, \ldots, 2m, \\ A_i = A_{2m+2-i}, \quad B_i = B_{2m+2-i}, \quad i = 1, 2, \ldots, m+1. \end{aligned}\right\} \tag{10.5}$$

Conditions (10.2) which are generalizations of conditions (2.2) for the symmetrically assembled multi-layer plates, were first used by Osternik and Barg [52] when they considered the theory of bending of multi-layer anisotropic plates including the effects of transverse shears.

Inverting the generalized Hooke's law equations (1.4.5) and considering assumption (c) the following expressions are obtained for the stresses in each layer:

$$\left.\begin{aligned} \sigma_x^i = B_{11}^i e_x + B_{12}^i e_y, \quad \sigma_y^i = B_{12}^i e_x + B_{22}^i e_y, \\ \tau_{xz}^i = B_{55}^i e_{xz}, \quad \tau_{yz}^i = B_{44}^i e_{yz}, \quad \tau_{xy}^i = B_{66}^i e_{xy}, \end{aligned}\right\} \tag{10.6}$$

where

$$\left.\begin{aligned} B_{11}^{i} &= \frac{a_{22}^{i}}{\Omega_i} = \frac{E_1^i}{1-\nu_1^i\nu_2^i}, \qquad B_{22}^{i} = \frac{a_{11}^{i}}{\Omega_i} = \frac{E_2^i}{1-\nu_1^i\nu_2^i}, \\ B_{12}^{i} &= -\frac{a_{12}^{i}}{\Omega_i} = \frac{\nu_2 E_1^i}{1-\nu_1^i\nu_2^i} = \frac{\nu_1^i E_2^i}{1-\nu_1^i\nu_2^i}, \\ B_{55}^{i} &= \frac{1}{a_{55}^{i}} = G_{13}^{i}, \qquad B_{44}^{i} = \frac{1}{a_{44}^{i}} = G_{23}^{i}, \qquad B_{66}^{i} = \frac{1}{a_{66}^{i}} = G_{12}^{i}, \end{aligned}\right\} \tag{10.7}$$

$$\Omega_i = a_{11}^{i} a_{22}^{i} - (a_{12}^{i})^2 = \frac{1-\nu_1^i\nu_2^i}{E_1^i E_2^i}.$$

From (1.2.2) on the basis of (10.1), (10.2) and (10.6)

$$\left.\begin{aligned} e_{xz}^{i} &= \frac{\partial u_x^i}{\partial z} + \frac{\partial w}{\partial x} = \frac{1}{G_{13}^{i}}\tau_{xz}^{i} = f(z)\varphi + \frac{A_i}{G_{13}^{i}}\varphi, \\ e_{yz}^{i} &= \frac{\partial u_y}{\partial z} + \frac{\partial w}{\partial y} = \frac{1}{G_{23}^{i}}\tau_{zy}^{i} = f(z)\psi + \frac{B_i}{G_{23}^{i}}\psi. \end{aligned}\right\} \tag{10.8}$$

Integration of equations (10.8) with respect to z from zero to z and considering (10.4) results in the following expressions for u_x^i and u_y^i:

$$\left.\begin{aligned} u_x^{i} &= -z\frac{\partial w}{\partial x} + J_0(z)\varphi + z\frac{A_i}{G_{13}^{i}}\varphi + R_1^{i}\varphi \operatorname{sign} z \\ u_y^{i} &= -z\frac{\partial w}{\partial y} + J_0(z)\psi + z\frac{B_i}{G_{23}^{i}}\psi + R_2^{i}\psi \operatorname{sign} z \end{aligned}\right\} \tag{10.9}$$

where

$$\left.\begin{aligned} R_1^{m+1} &= 0, \qquad R_2^{m+1} = 0, \\ R_1^{i} &= \sum_{r=m+1}^{i-1} h_r\left(\frac{A_r}{G_{13}^{r}} - \frac{A_{r+1}}{G_{13}^{r+1}}\right), \qquad i = m+2, \ldots, 2m+1, \\ R_2^{i} &= \sum_{r=m+1}^{i-1} h_r\left(\frac{B_r}{G_{23}^{r}} - \frac{B_{r+1}}{G_{23}^{r+1}}\right), \qquad i = m+2, \ldots, 2m+1, \end{aligned}\right\} \tag{10.10}$$

$$R_1^i = R_1^{2m+2-i}, \quad R_2^i = R_2^{2m+2-i}, \quad i = 1, 2, \ldots, m,$$

And for J_0

$$J_0 = \int_0^z f(z)\,dz. \tag{10.11}$$

Substituting values of u_x and u_y from (10.9) into (1.2.2) expressions are obtained for the strain components

$$\left.\begin{aligned}
e_x^i &= -z\frac{\partial^2 w}{\partial x^2} + \left[J_0 + \frac{A_i}{G_{13}^i}z + R_1^i \operatorname{sign} z\right]\frac{\partial \varphi}{\partial x},\\
e_y^i &= -z\frac{\partial^2 w}{\partial y^2} + \left[J_0 + \frac{B_i}{G_{23}^i}z + R_2^i \operatorname{sign} z\right]\frac{\partial \psi}{\partial y},\\
e_{xy}^i &= -2z\frac{\partial^2 w}{\partial x\,\partial y} + \left[J_0 + \frac{A_i}{G_{13}^i}z + R_1^i \operatorname{sign} z\right]\frac{\partial \varphi}{\partial y} +\\
&\quad + \left[J_0 + \frac{B_i}{G_{23}^i}z + R_2^i \operatorname{sign} z\right]\frac{\partial \psi}{\partial x}.
\end{aligned}\right\} \tag{10.12}$$

Substituting the strain values from (10.12) into (10.6) the following expressions are obtained for stresses:

$$\sigma_x^i = -z\left(B_{11}^i\frac{\partial^2 w}{\partial x^2} + B_{12}^i\frac{\partial^2 w}{\partial y^2}\right) + B_{11}^i\left(J_0 + \frac{A_i}{G_{13}^i}z + R_1^i \operatorname{sign} z\right)\frac{\partial \varphi}{\partial x} +$$

$$+ B_{12}^i\left(J_0 + \frac{B_i}{G_{23}^i}z + R_2^i \operatorname{sign} z\right)\frac{\partial \psi}{\partial y}, \tag{10.13}$$

$$\sigma_y^i = -z\left(B_{22}^i\frac{\partial^2 w}{\partial y^2} + B_{12}^i\frac{\partial^2 w}{\partial x^2}\right) + B_{22}^i\left(J_0 + \frac{B_i}{G_{23}^i}z + R_2^i \operatorname{sign} z\right)\frac{\partial \psi}{\partial y} +$$

$$+ B_{12}^i\left(J_0 + \frac{A_i}{G_{13}^i}z + R_1^i \operatorname{sign} z\right)\frac{\partial \varphi}{\partial x}, \tag{10.14}$$

$$\tau_{xy}^i = -2zB_{66}^i\frac{\partial^2 w}{\partial x\,\partial y} + B_{66}^i\left[\left(J_0 + \frac{A_i}{G_{13}^i}z + R_1^i \operatorname{sign} z\right)\frac{\partial \varphi}{\partial y} +\right.$$

$$\left. + \left(J_0 + \frac{B_i}{G_{23}^i}z + R_2^i \operatorname{sign} z\right)\frac{\partial \psi}{\partial x}\right]. \tag{10.15}$$

Internal bending and twisting moment resultants and transverse shear stress resultants are determined in the usual way [53]:

$$M_x = 2\int_0^{h_{m+1}} \sigma_x^{m+1} z\,dz + 2\sum_{s=1}^{m}\int_{h_{s+1}}^{h_s} \sigma_x^s z\,dz =$$

$$= -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} + P_{111}\frac{\partial \varphi}{\partial x} + P_{122}\frac{\partial \psi}{\partial y}, \quad (10.16)$$

$$M_y = 2\int_0^{h_{m+1}} \sigma_y^{m+1} z\,dz + 2\sum_{s=1}^{m}\int_{h_{s+1}}^{h_s} \sigma_y^s z\,dz =$$

$$= -D_{22}\frac{\partial^2 w}{\partial y^2} - D_{12}\frac{\partial^2 w}{\partial x^2} + P_{222}\frac{\partial \psi}{\partial y} + P_{121}\frac{\partial \varphi}{\partial x}, \quad (10.17)$$

$$H = 2\int_0^{h_{m+1}} \tau_{xy}^{m+1} z\,dz + 2\sum_{s=1}^{m}\int_{h_{s+1}}^{h_s} \tau_{xy}^s z\,dz =$$

$$= -2D_{66}\frac{\partial^2 w}{\partial x\,\partial y} + P_{661}\frac{\partial \varphi}{\partial y} + P_{662}\frac{\partial \psi}{\partial x}, \quad (10.18)$$

$$N_x = 2\int_0^{h_{m+1}} \tau_{xz}^{m+1} dz + 2\sum_{s=1}^{m}\int_{h_{s+1}}^{h_s} \tau_{xz}^s dz = K_1\varphi, \quad (10.19)$$

$$N_y = 2\int_0^{h_{m+1}} \tau_{yz}^{m+1} dz + 2\sum_{s=1}^{m}\int_{h_{s+1}}^{h_s} \tau_{yz}^s dz = K_2\psi, \quad (10.20)$$

The coefficients in (10.16) – (10.20) are defined as follows:

$$D_{jk} = \frac{2}{3}\left[B_{jk}^{m+1} h_{m+1}^3 + \sum_{s=1}^{m} B_{jk}^s (h_s^3 - h_{s+1}^3)\right], \quad (10.21)$$

$$\left.\begin{aligned}
P_{jk1} &= 2B_{jk}^{m+1}\left(J_7 + \frac{A_{m+1}}{G_{13}^{m+1}}\frac{h_{m+1}^3}{3}\right) + \\
&+ 2\sum_{s=1}^{m} B_{jk}^s\left[J_8 + \frac{A_s}{G_{13}^s}\frac{h_s^3 - h_{s+1}^3}{3} + \frac{R_1(h_s^2 - h_{s+1}^2)}{2}\right], \\
P_{jk2} &= 2B_{jk}^{m+1}\left(J_7 + \frac{B_{m+1}}{G_{23}^{m+1}}\frac{h_{m+1}^3}{3}\right) + \\
&+ 2\sum_{s=1}^{m} B_{jk}^s\left[J_8 + \frac{B_s}{G_{23}^s}\frac{h_s^3 - h_{s+1}^3}{3} + \frac{R_2^s(h_s^2 - h_{s+1}^2)}{2}\right],
\end{aligned}\right\} \quad (10.22)$$

$$\left.\begin{aligned} K_1 &= 2G_{13}^{m+1}J_9 + 2A_{m+1}h_{m+1} + 2\sum_{s=1}^{m}[G_{13}^{s}J_{10} + A_s(h_s - h_{s+1})], \\ K_2 &= 2G_{23}^{m+1}J_9 + 2B_{m+1}h_{m+1} + 2\sum_{s=1}^{m}[G_{23}^{s}J_{10} + B_s(h_s - h_{s+1})], \end{aligned}\right\} \quad (10.23)$$

where

$$\left.\begin{aligned} J_7 &= \int_0^{h_{m+1}} J_0 z\,dz, \qquad J_8 = \int_{h_{s+1}}^{h_s} J_0 z\,dz, \\ J_9 &= \int_0^{h_{m+1}} f\,dz, \qquad J_{10} = \int_{h_{s+1}}^{h_s} f\,dz. \end{aligned}\right\} \quad (10.24)$$

Substituting values of the internal stress and moment resultants from (10.16) – (10.20) into (2.47) – (2.49) ($X_1 = Y_1 = 0$), a complete system of equations for transverse bending of multi-layer orthotropic plates is obtained

$$K_1\frac{\partial\varphi}{\partial x} + K_2\frac{\partial\psi}{\partial y} = -Z_2, \quad (10.25)$$

$$D_{11}\frac{\partial^3 w}{\partial x^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial x\,\partial y^2} - P_{111}\frac{\partial^2\varphi}{\partial x^2} - P_{661}\frac{\partial^2\varphi}{\partial y^2} - (P_{122} + P_{662})\frac{\partial^2\psi}{\partial x\,\partial y} + K_1\varphi = 0, \quad (10.26)$$

$$D_{22}\frac{\partial^3 w}{\partial y^3} + (D_{12} + 2D_{66})\frac{\partial^3 w}{\partial y\,\partial x^2} - P_{222}\frac{\partial^2\psi}{\partial y^2} - P_{662}\frac{\partial^2\psi}{\partial x^2} - (P_{121} + P_{661})\frac{\partial^2\varphi}{\partial x\,\partial y} + K_2\psi = 0. \quad (10.27)$$

Comparison of the system of equations (10.25) – (10.27) with the corresponding system for the one-layer orthotropic plate (3.3) – (3.5) or (5.5) – (5.7) shows that the systems are completely identical except for different constant coefficients. As opposed to one-layer plates, the coefficients in the equations for the case of multi-layer plates are defined through the use of cumbersome formulas (10.21) – (10.23).

Stresses in individual layers of multi-layer plates are of importance for strength analysis, and are determined from equations (10.2), (10.13) – (10.15).

Finally, the boundary conditions for multi-layer plates can be constructed in a manner similar to the case of one-layer plates. (see Section 3). However, it should be remembered that boundary conditions derived in such a way should be carefully examined in each individual case because the problem of boundary

conditions, when considering bending of laminated anisotropic plates, including the effect of transverse shears, requires a cautious approach and supplementary studies.

When the function of transverse deformations f(z) is selected in the form of a parabola, i.e., according to formula (4.1), the integrals present in (10.11) – (10.23) are obtained from

$$\left.\begin{aligned} J_0 &= \frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right), \quad J_7 = \frac{h_{m+1}^3}{6}\left(\frac{h^2}{4}-\frac{h_{m+1}^2}{5}\right), \\ J_8 &= \frac{h^2}{24}(h_s^3 - h_{s+1}^3) - \frac{1}{30}(h_s^5 - h_{s+1}^5), \\ J_9 &= \frac{h_{m+1}}{2}\left(\frac{h^2}{4}-\frac{h_{m+1}^2}{3}\right), \\ J_{10} &= \frac{h^2}{8}(h_s - h_{s+1}) - \frac{1}{6}(h_s^3 - h_{s+1}^3). \end{aligned}\right\} \quad (10.28)$$

The subsequent chapters of this book consider the solutions of actual problems for one-layer and two-layer plates. In the case of multi-layer plates the reader should use the results obtained in this section and apply them appropriately because the solution procedures for one-layer and multi-layer plates are the same.

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CHAPTER III

DETERMINATION OF THE STATE OF STRESS AND DEFORMATION OF PLATES

1. Bending of a Plate Into a Cylindrical Surface

Consider a long rectangular plate with a transverse load which is constant along the plate length. The plate is uniformly supported along the long sides and arbitrarily supported along the short sides. The bent surface of the plate in this case, at a sufficient distance from the short sides, resembles a cylindrical surface.

As is usual [1] the origin of coordinates is taken on the long side away from the short sides, and the y-axis is directed along the long side (Figure 12.). We consider in this case that the functions u, v, w, φ, and ψ depend only on the x coordinate.

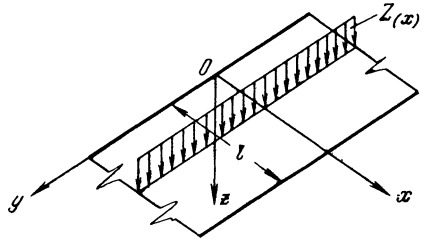

Figure 12.

1. Orthotropic plate. Consider an orthotropic plate with principal directions of elasticity which coincide with the directions of the coordinate axes x, y, z.

   For internal stress and moment resultants and for stresses we have for this case, from expressions (2.5.8) – (2.5.17):

$$\left.\begin{aligned} T_x &= C_{11}\frac{du}{dx} - hA_1Z_1, \quad S = C_{66}\frac{dv}{dx}, \\ T_y &= C_{12}\frac{du}{dx} - hA_2Z_1; \end{aligned}\right\} \tag{1.1}$$

$$\left.\begin{aligned} M_x &= -D_{11}\frac{d^2w}{dx^2} + \frac{h^2}{10}a_{55}D_{11}\frac{d\varphi}{dx} - \frac{h^2}{10}A_1Z_2, \\ M_y &= -D_{12}\frac{d^2w}{dx^2} + \frac{h^2}{10}a_{55}D_{12}\frac{d\varphi}{dx} - \frac{h^2}{10}A_2Z_2, \\ H &= \frac{h^2}{10}D_{66}a_{44}\frac{d\psi}{dx}; \end{aligned}\right\} \tag{1.2}$$

$$N_x = \frac{h^3}{12}\varphi, \qquad N_y = \frac{h^2}{12}\psi; \tag{1.3}$$

$$\left.\begin{aligned} \sigma_x &= B_{11}\left(\frac{du}{dx} - z\frac{d^2w}{dx^2}\right) + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)a_{55}B_{11}\frac{d\varphi}{dx} - \\ &\quad - A_1\left[Z_1 + \left(\frac{3}{2}\frac{z}{h} - 2\frac{z^3}{h^3}\right)Z_2\right], \\ \sigma_y &= B_{12}\left(\frac{du}{dx} - z\frac{d^2w}{dx^2}\right) + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)a_{55}B_{12}\frac{d\varphi}{dx} - \\ &\quad - A_2\left[Z_1 + \left(\frac{3}{2}\frac{z}{h} - 2\frac{z^3}{h^3}\right)Z_2\right], \\ \tau_{xy} &= B_{66}\left[\frac{dv}{dx} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)a_{44}\frac{d\psi}{dx}\right]. \end{aligned}\right\} \tag{1.4}$$

The functions u, v, w, φ, and ψ are determined from the system of equations (2.5.3) – (2.5.7) which for this case is:

$$C_{11}\frac{d^2u}{dx^2} = A_1h\frac{dZ_1}{dx}, \qquad C_{66}\frac{d^2v}{dx^2} = 0, \tag{1.5}$$

$$\frac{d\varphi}{dx} = -\frac{12}{h^3}Z_2, \tag{1.6}$$

$$D_{11}\frac{d^3w}{dx^3} - \frac{h^2}{10}a_{55}D_{11}\frac{d^2\varphi}{dx^2} + \frac{h^3}{12}\varphi = -A_1\frac{h^2}{10}\frac{dZ_2}{dx}, \tag{1.7}$$

$$\frac{d^2\psi}{dx^2} - k^2\psi = 0, \qquad k^2 = \frac{5h}{6a_{44}D_{66}}. \tag{1.8}$$

Examination of equations (1.5) – (1.8) reveals that here we have in fact two independent problems. The first is the trivial generalized plane stress problem which is described by equations (1.5), and it is assumed that the load Z_1 and the boundary conditions are such that u = 0, v = 0.

This problem is not considered here. The second problem is the transverse bending of the plate with the consideration of transverse shears and normal stresses σ_z and it is described by the system of equations (1.6) – (1.8) with respect to the three functions w(x), φ(x), and ψ(x).

It is assumed that the plate is of thickness h = constant and is bent by a uniformly distributed load applied in such a way that

$$Z^+ = \frac{q}{2}, \quad Z^- = \frac{q}{2}$$

so that in the notation (2.2.29)

$$Z_1 = 0, \quad Z_2 = q = \text{constant} \tag{1.9}$$

On the basis of these equalities from (1.6) we have for φ:

$$\varphi = -\frac{12}{h^3} qx - \frac{12}{h^3} c_1. \tag{1.10}$$

Substituting values of φ from (1.10) into (1.7) and considering (1.9) we obtain:

$$D_{11} \frac{d^3w}{dx^3} = qx + c_1, \tag{1.11}$$

and from this we obtain for w:

$$D_{11}w = q\frac{x^4}{24} + c_1\frac{x^3}{6} + c_2\frac{x^2}{2} + c_3x + c_4. \tag{1.12}$$

Finally, from (1.8) we can obtain the third function ψ:

$$\psi = c_5e^{hx} + c_6e^{-hx}. \tag{1.13}$$

Here $c_1 \ldots\ldots, c_6$ are integration constants.

The integration constants are determined from the boundary conditions.

Let the plate of width l be simply-supported along its long sides. Then the boundary conditions are,(2.3.14):

at	x = 0	w = 0	$M_x = 0$	H = 0	(1.14)
at	x = l	w = 0	$M_x = 0$	H = 0	

When the boundary conditions (1.14) are satisfied, all six integration constants c_i are determined. Substituting their values into (1.10), (1.12) and (1.13) results in

$$\varphi = \frac{12}{h^3} q \left(\frac{l}{2} - x \right), \quad \psi = 0, \tag{1.15}$$

$$w = \frac{q}{D_{11}} \left(\frac{x^4}{24} - \frac{x^3 l}{12} + \frac{l^3 x}{24} \right) + \left[\frac{3a_{55}}{5h} + \frac{A_1 h^2}{20 D_{11}} \right] (lx - x^2) q. \tag{1.16}$$

Now the stresses and moment resultants are found from (1.2) – (1.4) by substituting these values of φ, ψ, and w.

In equation (1.16) the first term represents the normal displacement of the plate which is found with the classical theory, and the second term is a correction to the first term; the first part of this correction results from the consideration of transverse shears, and the second, from the consideration of normal stresses σ_z.

For the deflection of the centerline of the plate, i.e., the line $x = l/2$, we obtain from equation (1.16)

$$w_c = \overline{w}_0 \left[1 + \frac{24}{25} \left(\frac{B_{11}}{B_{55}} + A_1 \right) \frac{h^2}{l^2} \right], \tag{1.17}$$

where

$$\overline{w}_0 = \frac{5ql^4}{384 D_{11}}$$

is the deflection of the centerline of the plate as found in the classical theory [2].

It is evident from (1.16) and (1.17) that the results for the normal displacements obtained from the classical theory, for certain values of the elastic constants of the plate material and of the relative plate thickness, could differ significantly from the corresponding values of normal displacements as obtained from the theory suggested in this book.

Furthermore, on the basis of (2.5.2), (1.16) and (1.17) it is not difficult to conclude that the consideration of normal stress σ_z amounts to a very small correction. This point is illustrated by the following equation for the normal displacement of the centerline for the case of an isotropic plate

$$w_c = \overline{w}_0 \left[1 + 2 \frac{24}{25} \frac{1+\nu}{1-\nu^2} \frac{h^2}{l^2} - \nu \frac{24}{25} \frac{1+\nu}{1-\nu^2} \frac{h^2}{l^2} \right] \tag{1.18}$$

where the second term within the brackets with a coefficient 2 represents the correction when the transverse shears are considered, and the third term with coefficient ν represents the correction due to normal stresses σ_z. In particular, when $h/l = 1/4$ and $\nu = 0.3$ the correction due to σ_z is 2.6%, while the correction due to transverse shear amounts to 17%.

The above statements are also valid for the other pertinent quantities, for example for the stresses, which are written as follows:

$$\left.\begin{aligned}
\sigma_x &= \frac{6q}{h^3}(lx - x^2)z - \left(\frac{3}{10}\frac{z}{h} - 2\frac{z^3}{h^3}\right)\left(\frac{B_{11}}{B_{55}} + A_1\right)q, \\
\sigma_y &= \frac{6q}{h^3}\frac{B_{12}}{B_{11}}(lx - x^2)z - \left(\frac{3}{10}\frac{z}{h} - 2\frac{z^3}{h^3}\right)\frac{B_{12}}{B_{55}}q + \\
&+ \frac{6q}{5h}\frac{R_{12}}{B_{11}}A_1 z - \left(\frac{3}{2}\frac{z}{h} - 2\frac{z^3}{h^3}\right)A_2 q, \qquad \tau_{xy} = 0.
\end{aligned}\right\} \tag{1.19}$$

Let one long side of the same plate be fixed (x = 0) and the other long side (x = l) be free and uniformly loaded by a transverse edge load of intensity Q.

The boundary conditions in this case upon consideration of (2.4.1), (2.3.15) and (2.3.13) are

$$\left.\begin{aligned}
&\text{at } x = 0 \qquad w = 0, \quad \psi = 0, \\
&-\frac{dw}{dx} + a_{55}\frac{1}{2}\left(\frac{h^2}{4} - \frac{z_0^2}{3}\right)\varphi = 0, \\
&\text{at } x = l \qquad N_x = Q, \quad M_x = 0, \quad H = 0.
\end{aligned}\right\} \tag{1.20}$$

Considering that $Z_1 = Z_2 = 0$ for this problem, we obtain from (1.6), (1.7) and (1.8), after integration, the following expressions

$$\left.\begin{aligned}
\varphi &= c_1 \quad \psi = c_5 e^{kx} + c_6 e^{-kx}, \\
D_{11}w &= -c_1\frac{h^2}{72}x^3 + c_2\frac{x^2}{2} + c_3 x + c_4,
\end{aligned}\right\} \tag{1.21}$$

where the c_i represent integration constants.

Using the boundary conditions (1.20) and substituting the values of the integration constants c_i into (1.21), we obtain the solutions for φ, ψ, and w:

$$\left.\begin{aligned}
\varphi &= \frac{12}{h^3}Q; \qquad \psi = 0, \\
w &= \frac{Q}{D_{11}}\left(\frac{x^2 l}{2} - \frac{x^3}{6}\right) + Qa_{55}\frac{x}{h}\left(\frac{3}{2} - 2\frac{z_0^2}{h^2}\right).
\end{aligned}\right\} \tag{1.22}$$

The other quantities of interest for this problem can be determined from (1.2) – (1.4).

In the expression (1.22) for w the first term represents the normal plate displacement which is found from the classical theory and which corresponds to the classical boundary conditions ($w = 0$, $\frac{\partial w}{\partial x} = 0$). The second term represents the correction due to the consideration of transverse shears.

Different variations in the clamping of the plate edges are obtained when z_0 is changed from zero to h/2 in the expression for w. For comparison sake the effect on the value of the maximum normal plate displacement due to three different assumptions for z_0 are given below.

Assuming that the maximum value of w is obtained as x = l, we have:

$$w_{\max} = \bar{w}_0 \left[1 + \frac{1}{4} \frac{B_{11}}{B_{55}} \frac{h^2}{l^2} \left(\frac{3}{2} - 2 \frac{z_0^2}{h^2} \right) \right]; \qquad (1.23)$$

where

$$\bar{w}_0 = \frac{Ql^3}{3D_{11}}$$

is the value of the maximum deflection of the plate as found with the classical theory.

The first variation of clamping. Let $z_0 = h/2$, then the boundary conditions at x = 0 are

$$w = 0, \quad \psi = 0, \quad -\frac{dw}{dx} + \frac{h^2}{12} a_{55} \varphi = 0, \qquad (1.24)$$

and for the corresponding w_{max}:

$$w_{\max} = \bar{w}_0 \left(1 + \frac{1}{4} \frac{B_{11}}{B_{44}} \frac{h^2}{l^2} \right). \qquad (1.25)$$

The second variation of clamping. Let

$$z_0 = \sqrt{\frac{3}{20}}\, h,$$

then the boundary conditions at x = 0 are

$$w = 0, \quad \psi = 0, \quad -\frac{dw}{dx} + \frac{h^2}{10} a_{55} \varphi = 0, \qquad (1.26)$$

which represent the Reissner boundary conditions [3, 4] in our notation. For w_{max} we obtain:

$$w_{\max} = \bar{w}_0 \left(1 + \frac{3}{10} \frac{B_{11}}{B_{55}} \frac{h^2}{l^2} \right). \qquad (1.27)$$

The third variation of clamping. Let $z_0 = 0$, then the boundary conditions at x = 0 are

$$w = 0, \quad \psi = 0, \quad -\frac{dw}{dx} + \frac{h^2}{8} a_{55}\varphi = 0, \tag{1.28}$$

which coincides with results obtained previously [5]. For w_{max} we have:

$$w_{max} = \bar{w}_0 \left(1 + \frac{3}{8}\frac{B_{11}}{B_{55}}\frac{h^2}{l^2}\right). \tag{1.29}$$

A comparison of (1.25), (1.27) and (1.29) reveals that they differ very little. For example, when

$$\frac{B_{11}}{B_{55}}\frac{h^2}{l^2} = 1$$

the maximum difference between the values of w_{max} is less than 10%. The difference between the first two values of w_{max}, which are the most important variations of clamping, amount to only 4%.

In each actual case it is possible to construct a mathematical model for the clamping by a corresponding selection of z_0, which would represent the clamping case under consideration. The error in the determination of the solution at a distance from the place of clamping will be within the accuracy of the determination of the actual clamping.

From (1.22) and (1.4) we obtain for stresses

$$\sigma_x = z(x - l)\frac{12}{h^3}Q, \quad \sigma_y = \nu_2\sigma_x, \quad \tau_{xy} = 0. \tag{1.30}$$

The values of w_{max} and of the stress σ_x agree with accurate values for the maximum displacement and normal stress of the isotropic rectangular cantilevered beam bent by a force Q. The accurate value of the shear stress τ_{xz} (as was expected [6]) starting from some distance from the clamped boundary ($\approx$ h/5) is almost identical with the value found in this book. The variation of this stress along the plate thickness (h) in the vicinity of the clamped boundary is not equivalent to the parabolic law accepted here.

During the evaluation of the state of stress at the plate boundary a careful approach is required not only in the case of the theory of Kirchhoff plates [7, 8], but also for the theory suggested here which may be used for the determination of the state of stress and deformation in the internal region of the plate.

2. A general case of anisotropy. Consider the cylindrical bending of a plate

for the general case of anisotropy when only one plane of elastic symmetry (which is parallel to the plate middle plane) is present at each point of the plate.

Considering the assumptions and identities (2.4.1), (2.4.2) and disregarding the stress σ_z in comparison with other stresses, i.e., assuming that $A_1 = A_2 = A_3 = 0$, we obtain the following initial differential equations from (2.3.1) – (2.3.4) (assuming also that $X^{\pm} = 0$, $Y^{\pm} = 0$):

$$\left.\begin{aligned} C_{11}\frac{d^2u}{dx^2} + C_{16}\frac{d^2v}{dx^2} &= 0, \\ C_{66}\frac{d^2v}{dx^2} + C_{16}\frac{d^2u}{dx^2} &= 0, \end{aligned}\right\} \tag{1.31}$$

$$\frac{d\varphi}{dx} = -\frac{12}{h^3}Z_2, \tag{1.32}$$

$$D_{11}\frac{d^3w}{dx^3} - \frac{h^2}{10}\left(P_1\frac{d^2\varphi}{dx^2} + P_2\frac{d^2\psi}{dx^2}\right) + \frac{h^3}{12}\varphi = 0, \tag{1.33}$$

$$D_{16}\frac{d^3w}{dx^3} - \frac{h^2}{10}\left(P_3\frac{d^2\psi}{dx^2} + P_4\frac{d^2\varphi}{dx^2}\right) + \frac{h^3}{12}\psi = 0, \tag{1.34}$$

where

$$\left.\begin{aligned} P_1 &= a_{55}D_{11} + a_{45}D_{16}, \quad & P_2 &= a_{45}D_{11} + a_{44}D_{16}, \\ P_3 &= a_{44}D_{66} + a_{45}D_{16}, \quad & P_4 &= a_{45}D_{66} + a_{55}D_{16}. \end{aligned}\right\} \tag{1.35}$$

Without considering the plane problem, let us deal now with the transverse cylindrical bending of the plate on the assumption that u = 0 and v = 0, i.e., let us solve equations (1.32) – (1.34) for given boundary and loading conditions.

In this case from (2.2.38) – (2.2.42) the stress and moment resultants are:

$$N_x = \frac{h^3}{12}\varphi, \qquad N_y = \frac{h^3}{12}\psi, \tag{1.36}$$

$$\left.\begin{aligned} M_x &= -D_{11}\frac{d^2w}{dx^2} + \frac{h^2}{10}\left(P_1\frac{d\varphi}{dx} + P_2\frac{d\psi}{dx}\right), \\ M_y &= -D_{12}\frac{d^2w}{dx^2} + \frac{h^2}{10}\left(P_5\frac{d\psi}{dx} + P_6\frac{d\varphi}{dx}\right), \\ H &= -D_{16}\frac{d^2w}{dx^2} + \frac{h^2}{10}\left(P_4\frac{d\varphi}{dx} + P_3\frac{d\psi}{dx}\right), \end{aligned}\right\} \tag{1.37}$$

where

$$P_5 = a_{45}D_{12} + a_{44}D_{26}, \qquad P_6 = a_{55}D_{12} + a_{45}D_{26}. \tag{1.35'}$$

The stresses for this problem as given in (2.2.30) - (2.2.32) can be rewritten as follows:

$$\left.\begin{aligned}
\sigma_x &= -zB_{11}\frac{d^2w}{dx^2} + \frac{z}{h}\left(\frac{3}{2} - 2\frac{z^2}{h^2}\right)\left(P_1\frac{d\varphi}{dx} + P_2\frac{d\psi}{dx}\right),\\
\sigma_y &= -zB_{12}\frac{d^2w}{dx^2} + \frac{z}{h}\left(\frac{3}{2} - 2\frac{z^2}{h^2}\right)\left(P_5\frac{d\psi}{dx} + P_6\frac{d\varphi}{dx}\right),\\
\tau_{xy} &= -zB_{16}\frac{d^2w}{dx^2} + \frac{z}{h}\left(\frac{3}{2} - 2\frac{z^2}{h^2}\right)\left(P_4\frac{d\varphi}{dx} + P_3\frac{d\psi}{dx}\right).
\end{aligned}\right\} \quad (1.38)$$

Just as in the case of the orthotropic plate we assume

$$Z^+ = \frac{q}{2}, \quad Z^- = \frac{q}{2}, \quad Z_1 = 0, \quad Z_2 = q = \text{const.} \quad (1.39)$$

Because of (1.39) we obtain from equation (1.32) by integration:

$$\varphi = -\frac{12}{h^3}qx + c_1, \quad (1.40)$$

Then equations (1.33) and (1.34) become:

$$D_{11}\frac{d^3w}{dx^3} - \frac{h^2}{10}P_2\frac{d^2\psi}{dx^2} = qx - \frac{h^3}{12}c_1,$$

$$D_{16}\frac{d^3w}{dx^3} - \frac{h^2}{10}P_3\frac{d^2\psi}{dx^2} + \frac{h^3}{12}\psi = 0.$$

When w is eliminated from this system we obtain:

$$\frac{d^2\psi}{dx^2} - k^2\psi = \frac{10}{h^2}\frac{D_{16}}{P}\left(qx - \frac{h^3}{12}c_1\right), \quad (1.41)$$

where

$$P = a_{44}(D_{11}D_{66} - D_{16}^2), \qquad k^2 = \frac{5}{6}\frac{D_{11}}{P}h. \quad (1.42)$$

Integration of (1.41) leads to the solution for the function ψ:

$$\psi = c_2e^{kx} + c_3e^{-kx} - \frac{D_{16}}{D_{11}}\left(\frac{12}{h^3}qx - c_1\right), \quad (1.43)$$

For the determination of the normal displacement w the following differential equation is then obtained:

$$D_{11}\frac{d^3w}{dx^3} = \frac{h^2}{10}P_2k^2(c_2e^{kx} + c_3e^{-kx}) + qx - \frac{h^3}{12}c_1,$$

which produces

$$D_{11}w = \frac{qx^4}{24} - \frac{h^3}{12}c_1\frac{x^3}{6} + c_4\frac{x^2}{2} + c_5x + c_6 + \frac{h^2}{10}\frac{P_2}{k}(c_2e^{kx} - c_3e^{-kx}). \quad (1.44)$$

The integration constants c_i which are present in (1.40), (1.43) and (1.44) are determined from the boundary conditions of the problem.

Let the plate be of width l and simply-supported along the long sides; then the boundary conditions are (see (2.3.13)):

$$\left.\begin{array}{llll} x = 0 & w = 0, & M_x = 0, & H = 0; \\ x = l & w = 0, & M_x = 0, & H = 0. \end{array}\right\} \tag{1.45}$$

By satisfying the boundary conditions six integration constants c_i are determined, and when they are substituted into (1.40), (1.43), and (1.44) we obtain

$$w = \frac{qx^4}{24D_{11}} - \frac{qlx^3}{12D_{11}} + \frac{ql^3x}{24D_{11}} - \frac{3q}{5hD_{11}^2}(P_1D_{11} + P_2D_{16})(x^2 - xl) - \\ - \frac{6qP_2^2}{5hk^2a_{44}D_{11}^2}\left\{1 - \frac{1}{e^{kl}}[e^{kx} + e^{k(l-x)}]\right\}, \tag{1.46}$$

$$\left.\begin{array}{l} \varphi = \dfrac{6q}{h^3}(l - 2x), \\ \psi = \dfrac{D_{16}}{D_{11}}\dfrac{6q}{h^3}(l - 2x) + \dfrac{12P_2q}{h^3ka_{44}D_{11}}\dfrac{e^{kx} - e^{k(l-x)}}{e^{kl}}. \end{array}\right\} \tag{1.47}$$

All other pertinent quantities for the problem are easily found from (1.36) – (1.38) since w, φ and ψ are known.

Consider now a plate fixed along one long side (x = 0) with the other long side completely free (x = l), carrying a uniformly distributed transverse edge load of intensity Q.

The boundary conditions of the problem, (2.3.15) and (2.3.13), in this case because of (2.4.1) are

$$\left.\begin{array}{l} \text{at } x = 0 \quad w = 0, \quad a_{44}\psi + a_{45}\varphi = 0, \\ -\dfrac{dw}{dx} + \dfrac{1}{2}\left(\dfrac{h^2}{4} - \dfrac{z_0^2}{3}\right)(a_{55}\varphi + a_{45}\psi) = 0, \\ \text{at } x = 1 \quad N_x = Q, \quad M_x = 0, \quad H = 0. \end{array}\right\} \tag{1.48}$$

Considering that $Z_1 = Z_2 = 0$, and proceeding just as in the preceding problem we obtain:

$$\varphi = c_1, \quad \psi = c_2e^{kx} + c_3e^{-kx} + c_1\frac{D_{16}}{D_{11}},$$

$$D_{11}w = -c_1\frac{h^3x^3}{72} + c_4\frac{x^2}{2} + c_5x + c_6 + \frac{P_2h^2}{10k}(c_2e^{kx} - c_3e^{-kx}).$$

After the integration constants c_i are determined from (1.48) we obtain the following expressions for φ, ψ, and w:

$$\varphi = \frac{12}{h^3} Q, \tag{1.49}$$

$$\psi = \left[\frac{D_{16}}{D_{11}} - \frac{e^{k(l-x)} + e^{-k(l-x)}}{e^{kl}}\left(\frac{D_{16}}{D_{11}} + \frac{a_{45}}{a_{44}}\right)\right] \frac{12}{h^3} Q, \tag{1.50}$$

$$w = \left(\frac{1}{2}x^2 l - \frac{1}{6}x^3\right)\frac{Q}{D_{11}} + \\ + \frac{x}{h}\left[\frac{6}{5}\frac{P_2}{D_{11}}\left(\frac{D_{16}}{D_{11}} + \frac{a_{45}}{a_{44}}\right) + \frac{a_{44}a_{55} - a_{45}^2}{a_{44}}\left(\frac{3}{2} - 2\frac{z_0^2}{h^2}\right)\right] Q - \\ - \frac{6}{5hk}\frac{P_2}{D_{11}}\left(\frac{D_{16}}{D_{11}} + \frac{a_{45}}{a_{44}}\right)\left[1 - \frac{e^{k(l-x)} - e^{-k(l-x)}}{e^{kl}}\right] Q. \tag{1.51}$$

The product kL usually is greater than one. Because of this fact the formulas obtained here are somewhat simplified.

As in the previous solutions, the other pertinent quantities may be determined from equations (1.36) – (1.38).

3. Particular theory. The problem of the cylindrical bending of a plate for a more general case of anisotropy is considered here with the use of the equations of the particular theory presented in the preceding chapter, Section 6.

The initial equation (2.6.16) in this case is

$$D_{11}\frac{d^4 w}{dx^4} = Z_2(x) + \frac{h^2}{10}\left(P_1\frac{d^3\varphi_0}{dx^3} + P_2\frac{d^3\psi_0}{dx^3}\right) - \frac{h^2}{10}A_1\frac{d^2 Z_2(x)}{dx^2} \tag{1.52}$$

where from (2.6.3) and (2.6.4) we have for φ_0 and ψ_0:

$$\varphi_0 = -B_{11}\frac{d^3 w_0}{dx^3}, \qquad \psi_0 = -B_{16}\frac{d^3 w_0}{dx^3}, \tag{1.53}$$

and w_0 is determined from the equation of the classical theory for a plate bent into the cylindrical surface [9]:

$$D_{11}\frac{d^4 w_0}{dx^4} = Z_2(x). \tag{1.54}$$

On the basis of (2.6.8) – (2.6.13) we have equations (1.37) and (1.38) for the stresses and moment resultants in which known values φ_0 and ψ_0 (as determined from (1.5.3)) are introduced instead of φ and ψ.

The transverse shear resultants are determined from (2.6.14) and (2.6.15):

$$\left.\begin{aligned} N_x &= -D_{11}\frac{d^3w}{dx^3} + \frac{h^2}{10}\left[P_1\frac{d^2\varphi_0}{dx^2} + P_2\frac{d^2\psi_0}{dx^2} - A_1\frac{dZ_2}{dx}\right] \\ N_y &= -D_{16}\frac{d^3w}{dx^3} + \frac{h^2}{10}\left[P_3\frac{d^2\psi_0}{dx^2} + P_4\frac{d^2\varphi_0}{dx^2} - A_3\frac{dZ_2}{dx}\right] \end{aligned}\right\} \qquad (1.55)$$

The solution of the problem utilizing the particular theory follows the following course. Expressions for φ_0 and ψ_0 are found from the solution of equation (1.54) with the use of (1.53):

$$\varphi_0 = -\frac{12}{h^3}\int Z_2 dx - \frac{12}{h^3}c_1^0, \qquad \psi_0 = \frac{B_{16}}{B_{11}}\varphi_0, \qquad (1.56)$$

where the integration constant c_i^0 does not need to be determined because in all succeeding equations and formulas φ^0 and ψ^0 are present in the form of derivatives with respect to x.

When the values of φ^0 and ψ^0 from (1.56) are substituted into (1.52) the following final equation is obtained:

$$D_{11}\frac{d^4w}{dx^4} = Z_2 - \frac{h^2}{10}\left[\frac{12}{h^3}\left(P_1 + \frac{B_{16}}{B_{11}}P_2\right) + A_1\right]\frac{d^2Z_2}{dx^2}, \qquad (1.57)$$

which differs from the corresponding equation of the classical theory (1.54) only on the right hand side.

A comparison of equation (1.57) with the system of equations of the general theory (1.32) – (1.34) reveals their principal difference, namely, that the order of the system (1.32) – (1.34) is higher than that of equation (1.57) by two orders, i.e., in the case of the general theory the governing equation of the problem is of the sixth order while equation (1.57) is of fourth order. This is easily seen from expressions (1.40) – (1.44) and equation (1.57).

In order to demonstrate the method of solution for the case of the particular theory, we consider cylindrical bending of a plate simply-supported along its long sides and loaded by a uniformly distributed load of intensity q.

In this case $Z_2 = q$ = constant, and the boundary conditions are of the type (2.6.18):

$$\left.\begin{aligned} &\text{at } x = 0 \quad w = 0, \quad M_x = 0, \\ &\text{at } x = l \quad w = 0, \quad M_x = 0. \end{aligned}\right\} \qquad (1.58)$$

Substituting the value of Z_2 into (1.57) reduces the problem to the solution of the equation

$$D_{11}\frac{d^4w}{dx^4} = q \tag{1.59}$$

with the boundary conditions (1.58), where on the basis of (2.6.8), (1.37) and (1.56) we have

$$M_x = -D_{11}\frac{d^2w}{dx^2} - \frac{6}{5h}\left(P_1 + P_2\frac{B_{16}}{B_{11}}\right)q. \tag{1.60}$$

Solving equation (1.59) and determining four integration constants from the boundary conditions (1.58) a final expression for w is obtained:

$$w = \frac{qx^4}{24D_{11}} - \frac{qlx^3}{12D_{11}} + \frac{ql^3x}{24D_{11}} - \frac{3q}{5hD_{11}^2}(P_1D_{11} + P_2D_{16})(x^2 - xl). \tag{1.61}$$

This expression differs from the corresponding one for w in the general theory, i.e., from (1.46), by the last term in (1.46) which appears due to the consideration of transverse shear.

In the particular case of an orthotropic plate when the principal directions of elasticity coincide with the principal geometric directions (x = constant, y = constant), we obtain with the use of (1.61) the following expression for w:

$$w = \frac{qx^4}{24D_{11}} - \frac{qlx^3}{12D_{11}} + \frac{ql^3x}{24D_{11}} - \frac{3q}{5h}a_{55}(x^2 - xl), \tag{1.62}$$

As could be expected, this expression is identical to that found from the general theory, namely with (1.16), with $A_1 = 0$.

2. Bending of a Simply-Supported Orthotropic Rectangular Plate Under a Sinusoidal Load

Consider an orthotropic rectangular plate of dimensions a, b, simply-supported along the entire contour, which is bent by a normally applied load (X = 0, Y = 0) distributed according to

$$\left.\begin{aligned} Z_1 &= 0, \\ Z_2 &= q\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \end{aligned}\right\} \tag{2.1}$$

where q is the load intensity at the center of the plate (x = a/2, y = b/2). See Figure 13.

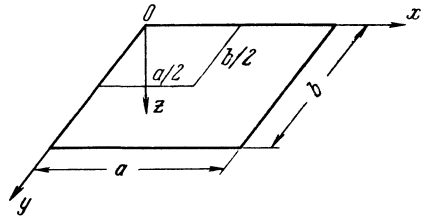

Figure 13.

The system of differential equations for bending of an orthotropic plate (2.5.5) – (2.5.7) is in this case

$$\left.\begin{aligned}
&\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y}=-\frac{12}{h^3}q\sin\frac{\pi x}{a}\sin\frac{\pi y}{b},\\
&D_{11}\frac{\partial^3 w}{\partial x^3}+(D_{12}+2D_{66})\frac{\partial^3 w}{\partial x\,\partial y^2}-\frac{h^2}{10}\Big[a_{55}\Big(D_{11}\frac{\partial^2}{\partial x^2}+D_{66}\frac{\partial^2}{\partial y^2}\Big)\varphi+\\
&+a_{44}(D_{12}+D_{66})\frac{\partial^2\psi}{\partial x\,\partial y}\Big]+\frac{h^3}{12}\varphi=-qA_1\frac{h^2}{10}\frac{\pi}{a}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b},\\
&D_{22}\frac{\partial^3 w}{\partial y^3}+(D_{12}+2D_{66})\frac{\partial^3 w}{\partial y\,\partial x^2}-\frac{h^2}{10}\Big[a_{44}\Big(D_{22}\frac{\partial^2}{\partial y^2}+D_{66}\frac{\partial^2}{\partial x^2}\Big)\psi+\\
&+a_{55}(D_{12}+D_{66})\frac{\partial^2\varphi}{\partial x\,\partial y}\Big]+\frac{h^3}{12}\psi=-qA_2\frac{h^3}{10}\frac{\pi}{b}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}
\end{aligned}\right\}\qquad(2.2)$$

These equations described only the transverse bending of the plate. The complete system of equations for orthotropic plates contains equations (2.5.5) – (2.5.7), plus two additional equations (2.5.3) and (2.5.4), which describe the plane stress problem of the plate and are not of interest here.

Assuming that

$$\left.\begin{aligned}
&w=f\sin\frac{\pi x}{a}\sin\frac{\pi y}{b},\\
&\varphi=c_\varphi\cos\frac{\pi x}{a}\sin\frac{\pi y}{b},\qquad \psi=c_\psi\sin\frac{\pi x}{a}\cos\frac{\pi y}{b},
\end{aligned}\right\}\qquad(2.3)$$

the boundary conditions of simple supports (2.3.14′) are satisfied which in this case, on the bases of (2.4.1) and (2.4.2) are

$$\left.\begin{aligned}
&\text{at } x=0,\ x=a\\
&\qquad M_x=0,\quad w=0,\quad \psi=0;\\
&\text{at } y=0,\ y=b\\
&\qquad M_y=0,\quad w=0,\quad \varphi=0.
\end{aligned}\right\}\qquad(2.4)$$

Substituting the expressions for w, φ, and ψ from (2.3) into the governing equations (2.2), we obtain expressions for the coefficients f, c_φ, and c_ψ

$$f = \frac{\Delta_1}{\Delta_0} q, \qquad c_\varphi = \frac{\Delta_2}{\Delta_0} q, \qquad c_\psi = \frac{\Delta_3}{\Delta_0} q, \tag{2.5}$$

where the Δ_i are:

$$\Delta_0 = \frac{h^3}{12}\left[D_{11}\frac{\pi^4}{a^4} + 2(D_{12}+2D_{66})\frac{\pi^4}{a^2b^2} + D_{22}\frac{\pi^4}{b^4}\right] + \\ + \frac{h^2}{10}\left[\left(D_{11}\frac{\pi^2}{a^2} + D_{66}\frac{\pi^2}{b^2}\right)\left(D_{22}\frac{\pi^2}{b^2} + D_{66}\frac{\pi^2}{a^2}\right) - \right. \\ \left. - (D_{12}+D_{66})^2\frac{\pi^4}{a^2b^2}\right]\left(a_{44}\frac{\pi^2}{a^2} + a_{55}\frac{\pi^2}{b^2}\right), \tag{2.6}$$

$$\Delta_1 = \frac{12}{h^3}\left\{\frac{h^6}{144} + \frac{h^5}{120}\left[a_{55}\left(D_{11}\frac{\pi^2}{a^2} + D_{66}\frac{\pi^2}{b^2}\right) + \right.\right. \\ \left. + a_{44}\left(D_{22}\frac{\pi^2}{b^2} + D_{66}\frac{\pi^2}{a^2}\right)\right] - \frac{h^4}{100}a_{44}a_{55}\left[(D_{12}+D_{66})\frac{\pi^4}{a^2b^2} - \right. \\ \left.\left. - \left(D_{11}\frac{\pi^2}{a^2} + D_{66}\frac{\pi^2}{b^2}\right)\left(D_{22}\frac{\pi^2}{b^2} + D_{66}\frac{\pi^2}{a^2}\right)\right]\right\} + \\ + \frac{h^2}{10}\left\{A_1\left[\frac{h^3}{12}\frac{\pi^2}{a^2} + a_{44}\frac{h^2}{10}\frac{\pi^2}{a^2}\left(D_{22}\frac{\pi^2}{b^2} + D_{66}\frac{\pi^2}{a^2}\right) - \right.\right. \\ \left. - a_{55}\frac{h^2}{10}(D_{12}+D_{66})\frac{\pi^4}{a^2b^2}\right] + A_2\left[\frac{h^3}{12}\frac{\pi^2}{b^2} + \right. \\ \left.\left. + a_{55}\frac{h^2}{10}\frac{\pi^2}{b^2}\left(D_{11}\frac{\pi^2}{a^2} + D_{66}\frac{\pi^2}{b^2}\right) - a_{44}\frac{h^2}{10}(D_{12}+D_{66})\frac{\pi^4}{a^2b^2}\right]\right\}, \tag{2.7}$$

$$\Delta_2 = \frac{12}{h^3}\frac{\pi}{a}\left\{\frac{h^3}{12}\left[D_{11}\frac{\pi^2}{a^2} + (D_{12}+2D_{66})\frac{\pi^2}{b^2}\right] + \right. \\ + \frac{h^2}{10}a_{44}\left[\left(D_{11}\frac{\pi^2}{a^2} + D_{66}\frac{\pi^2}{b^2}\right)\left(D_{22}\frac{\pi^2}{b^2} + D_{66}\frac{\pi^2}{a^2}\right) - \right. \\ \left.\left. - (D_{12}+D_{66})^2\frac{\pi^2}{a^2b^2}\right]\right\} - \frac{h^2}{10}\frac{\pi^3}{ab^2}\left\{A_1\left[(D_{12}+2D_{66})\frac{\pi^2}{a^2} + D_{22}\frac{\pi^2}{b^2}\right] - \right. \\ \left. - A_2\left[(D_{12}+2D_{66})\frac{\pi^2}{b^2} + D_{11}\frac{\pi^2}{a^2}\right]\right\}, \tag{2.8}$$

$$\Delta_3 = \frac{12}{h^3}\frac{\pi}{b}\left\{\frac{h^3}{12}\left[D_{22}\frac{\pi^2}{b^2}+(D_{12}+2D_{66})\frac{\pi^2}{a^2}\right]+\right.$$
$$+\frac{h^2}{10}a_{55}\left[\left(D_{11}\frac{\pi^2}{a^2}+D_{66}\frac{\pi^2}{b^2}\right)\left(D_{22}\frac{\pi^2}{b^2}+D_{66}\frac{\pi^2}{a^2}\right)-\right.$$
$$\left.\left.-(D_{12}+D_{66})^2\frac{\pi^4}{a^2b^2}\right]\right\}-\frac{h^2}{10}\frac{\pi^3}{ba^2}\left\{A_2\left[(D_{12}+2D_{66})\frac{\pi^2}{b^2}+D_{11}\frac{\pi^2}{a^2}\right]-\right.$$
$$\left.-A_1\left[(D_{12}+2D_{66})\frac{\pi^2}{a^2}+D_{22}\frac{\pi^2}{b^2}\right]\right\}, \quad (2.9)$$

Substituting the expressions for the functions w, φ, and ψ into (2.5.11)–(2.5.17) we obtain

$$M_x = \left[\left(D_{11}\frac{\pi^2}{a^2}+D_{12}\frac{\pi^2}{b^2}\right)\Delta_1-\right.$$
$$\left.-\frac{h^2}{10}\left(a_{55}D_{11}\Delta_2\frac{\pi}{a}+a_{44}D_{12}\Delta_3\frac{\pi}{b}+A_1\Delta_0\right)\right]\frac{q}{\Delta_0}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b},$$
$$M_y = \left[\left(D_{22}\frac{\pi^2}{b^2}+D_{12}\frac{\pi^2}{a^2}\right)\Delta_1-\right.$$
$$\left.-\frac{h^2}{10}\left(a_{44}D_{22}\Delta_3\frac{\pi}{b}+a_{55}D_{12}\Delta_2\frac{\pi}{a}+A_2\Delta_0\right)\right]\frac{q}{\Delta_0}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b},$$
$$H = \left[-2D_{66}\Delta_1\frac{\pi^2}{ab}+\frac{h^2}{10}D_{66}\left(a_{55}\Delta_2\frac{\pi}{b}+a_{44}\Delta_3\frac{\pi}{a}\right)\right]\frac{q}{\Delta_0}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b},$$
$$N_x = \frac{qh^3}{12}\frac{\Delta_2}{\Delta_0}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}, \quad N_y = \frac{qh^3}{12}\frac{\Delta_3}{\Delta_0}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b},$$
$$\sigma_x = \left[z\left(B_{11}\frac{\pi^2}{a^2}+B_{12}\frac{\pi^2}{b^2}\right)\Delta_1-\right.$$
$$-\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\left(a_{55}B_{11}\Delta_2\frac{\pi}{a}+a_{44}B_{12}\Delta_3\frac{\pi}{b}\right)-$$
$$\left.-\frac{z}{h}\left(\frac{3}{2}-2\frac{z^2}{h^2}\right)A_1\Delta_0\right]\frac{q}{\Delta_0}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b},$$
$$\sigma_y = \left[z\left(B_{22}\frac{\pi^2}{b^2}+B_{12}\frac{\pi^2}{a^2}\right)\Delta_1-\right.$$
$$-\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\left(a_{44}B_{22}\Delta_3\frac{\pi}{b}+a_{55}B_{12}\Delta_2\frac{\pi}{a}\right)-$$
$$\left.-\frac{z}{h}\left(\frac{3}{2}-2\frac{z^2}{h^2}\right)A_2\Delta_0\right]\frac{q}{\Delta_0}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b},$$

$$\tau_{xy} = \Big[-2zB_{66}\Delta_1 \frac{\pi^2}{ab} + \\ + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(a_{55}\Delta_2 \frac{\pi}{b} + a_{44}\Delta_3 \frac{\pi}{a}\right)\Big] \frac{q}{\Delta_0} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b}.$$

We consider now some particular cases.

1. Transversely isotropic plate. In the particular case of the transversely isotropic plate when the isotropy plane at each point of the plate is parallel to the middle plane, the elastic constants of the plate material on the basis of (1.4.10), (1.4.11), (2.5.1) and (2.5.2) are:

$$\left.\begin{aligned} &B_{11} = B_{22} = B_{12} + 2B_{66} = \frac{E}{1-\nu^2}, \quad B_{12} = \frac{\nu E}{1-\nu^2}, \\ &B_{66} = G = \frac{E}{2(1+\nu)}, \quad B_{12} + B_{66} = \frac{E}{2(1-\nu)}, \\ &a_{55} = a_{44} = \frac{1}{G'}, \quad a_{13} = a_{23} = -\frac{\nu_{13}}{E_3} = -\frac{\nu_{23}}{E_3} = -\frac{\nu'}{E'}, \\ &A_1 = A_2 = \\ &\quad = a_{13}(B_{11} + B_{12}) = a_{23}(B_{22} + B_{12}) = -\frac{\nu' E}{E'(1-\nu)}. \end{aligned}\right\} \quad (2.10)$$

The system of differential equations for bending (2.5.5) – (2.5.7) will be in this case:

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} = -\frac{12}{h^3} Z, \quad (2.11)$$

$$D\frac{\partial}{\partial x}\Delta w - \frac{h^2}{10}\frac{D}{G'}\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 \varphi}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 \psi}{\partial x\,\partial y}\right) + \frac{h^3}{12}\varphi = \\ = \frac{E}{E'}\frac{\nu'}{(1-\nu)}\frac{h^2}{10}\frac{\partial Z}{\partial x}, \quad (2.12)$$

$$D\frac{\partial}{\partial y}\Delta w - \frac{h^2 D}{10G'}\left(\frac{\partial^2 \psi}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 \psi}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 \varphi}{\partial x\,\partial y}\right) + \frac{h^3}{12}\psi = \\ = \frac{E}{E'}\frac{\nu'}{(1-\nu)}\frac{h^2}{10}\frac{\partial Z}{\partial y}, \quad (2.13)$$

where it was assumed that

$$Z_1 = 0, \quad Z_2 = Z, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (2.14)$$

The internal stress and moment resultants and the stresses, with the results (2.10), are:

$$\left.\begin{aligned} M_x &= -D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)+\frac{h^5}{60(1-\nu)}\frac{G}{G'}\left(\frac{\partial\varphi}{\partial x}+\nu\frac{\partial\psi}{\partial y}\right)+ \\ &\quad +\frac{\nu' E}{E'(1-\nu)}\frac{h^2}{10}Z, \\ M_y &= -D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)+\frac{h^5}{60(1-\nu)}\frac{G}{G'}\left(\frac{\partial\psi}{\partial y}+\nu\frac{\partial\varphi}{\partial x}\right)+ \\ &\quad +\frac{\nu' E}{E'(1-\nu)}\frac{h^2}{10}Z, \\ H &= -D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y}+\frac{h^5}{120}\frac{G}{G'}\left(\frac{\partial\varphi}{\partial y}+\frac{\partial\psi}{\partial x}\right), \\ N_x &= \frac{h^3}{12}\varphi, \qquad N_y = \frac{h^3}{12}\psi, \end{aligned}\right\} \tag{2.15}$$

$$\sigma_x = -z\bar{B}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\frac{\bar{B}}{G'}\left(\frac{\partial\varphi}{\partial x}+\nu\frac{\partial\psi}{\partial y}\right)+$$
$$+\left(\frac{3}{2}\frac{z}{h}-2\frac{z^3}{h^3}\right)\frac{\nu' E}{E'(1-\nu)}Z, \tag{2.16}$$

$$\left.\begin{aligned} \sigma_y &= -z\bar{B}\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\frac{\bar{B}}{G'}\left(\frac{\partial\psi}{\partial y}+\nu\frac{\partial\varphi}{\partial x}\right)+ \\ &\quad +\left(\frac{3}{2}\frac{z}{h}-2\frac{z^3}{h^3}\right)\frac{\nu' E}{E'(1-\nu)}Z, \\ \tau_{xy} &= G\left[-2z\frac{\partial^2 w}{\partial x\,\partial y}+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\frac{1}{G'}\left(\frac{\partial\varphi}{\partial y}+\frac{\partial\psi}{\partial x}\right)\right], \end{aligned}\right\} \tag{2.16}$$

where, as before,

$$\bar{B} = \frac{E}{1-\nu^2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}. \tag{2.17}$$

On the basis of (2.10), (2.3), and (2.5) – (2.9) the solution for the transversely isotropic rectangular plate can be written:

$$w = w_0\left[1+\frac{\pi^2 h^2}{10(1-\nu)}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(2\frac{G}{G'}-\nu'\frac{E}{E'}\right)\right], \tag{2.18}$$

$$\varphi = \frac{12ab^2 q}{\pi h^3(a^2+b^2)}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}, \tag{2.19}$$

$$\psi = \frac{12ba^2q}{\pi h^3(a^2+b^2)} \sin\frac{\pi x}{a}\cos\frac{\pi y}{b}, \tag{2.20}$$

where the normal displacement w_0 is the same as found from the classical theory for the transverse bending of a plate [10]

$$w_0 = \frac{12(1-\nu^2)a^4b^4q}{\pi^4Eh^3(a^2+b^2)^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}.$$

With these available expressions for the functions w, φ, and ψ, equations for the moment resultants and stresses are immediately obtained:

$$M_x = \left[DA\left(\frac{\pi^2}{a^2}+\nu\frac{\pi^2}{b^2}\right) - \frac{h^5}{60(1-\nu)}\frac{G}{G'}\left(B\frac{\pi}{a}+\nu C\frac{\pi}{b}\right) + \right.$$
$$\left. + \frac{\nu' E}{E'(1-\nu)}\frac{h^2}{10}q\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b},$$

$$M_y = \left[DA\left(\frac{\pi^2}{b^2}+\nu\frac{\pi^2}{a^2}\right) - \frac{h^5}{60(1-\nu)}\frac{G}{G'}\left(C\frac{\pi}{b}+\nu B\frac{\pi}{a}\right) + \right.$$
$$\left. + \frac{\nu' E}{E'(1-\nu)}\frac{h^2}{10}q\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b},$$

$$H = -\left[DA(1-\nu)\frac{\pi^2}{ab} - \frac{h^5}{120}\frac{G}{G'}\left(B\frac{\pi}{b}+C\frac{\pi}{a}\right)\right]\cos\frac{\pi x}{a}\cos\frac{\pi y}{b},$$

$$N_x = \frac{h^3}{12}B\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}, \qquad N_y = \frac{h^3}{12}C\sin\frac{\pi x}{a}\cos\frac{\pi y}{b},$$

$$\sigma_x = \left[z\bar{B}A\left(\frac{\pi^2}{a^2}+\nu\frac{\pi^2}{b^2}\right) - \frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\frac{\bar{B}}{G'}\left(B\frac{\pi}{a}+\nu C\frac{\pi}{b}\right) + \right.$$
$$\left. + \left(\frac{3}{2}\frac{z}{h}-2\frac{z^3}{h^3}\right)\frac{\nu' E}{E'(1-\nu)}q\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}.$$

$$\sigma_y = \left[z\bar{B}A\left(\frac{\pi^2}{b^2}+\nu\frac{\pi^2}{a^2}\right) - \frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\frac{\bar{B}}{G'}\left(C\frac{\pi}{b}+\nu B\frac{\pi}{a}\right) + \right.$$
$$\left. + \left(\frac{3}{2}\frac{z}{h}-2\frac{z^3}{h^3}\right)\frac{\nu' E}{E'(1-\nu)}q\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b},$$

$$\tau_{xy} = G\left[-2zA\frac{\pi^2}{ab} + \frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)\frac{1}{G'}\left(B\frac{\pi}{b}+C\frac{\pi}{a}\right)\right]\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}.$$

In the particular case of a square plate (a = b) the deflection of the center of the plate (x = a/2, y = b/2) can be obtained from (2.18):

$$w_c = \bar{w}_0\left[1 + \frac{\pi^2}{5(1-\nu)}\frac{h^2}{a^2}\left(2\frac{G}{G'}-\nu'\frac{E}{E'}\right)\right], \tag{2.21}$$

where

$$\bar{w}_0 = 3(1 - \nu^2)a^4q/\pi^4Eh^3.$$

Let us now attempt to find a solution for this case with the use of a theory which disregards the effect of the normal stress σ_z but includes the other accepted general assumptions, i.e., with the use of formulas (2.5.18) – (2.5.31), which differ from the equations used previously by the absence of terms with multipliers A_1 and A_2.

A derivation for the case of a transversely isotropic plate requires the elimination of terms from the final expressions for the functions φ, ψ, and w which contain the multiplier ν'. Then obviously equations (2.19) and (2.20) will not change and for normal displacement we have from (2.18):

$$w = w_0\left[1 + \frac{\pi^2h^2}{5(1-\nu)}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\frac{G}{G'}\right]. \qquad (2.22)$$

For deflection of the center of a square plate

$$w_c = \bar{w}_0\left[1 + \frac{2\pi^2}{5(1-\nu)}\frac{h^2}{a^2}\frac{G}{G'}\right]. \qquad (2.23)$$

It is evident from (2.18) – (2.23) that, just as before, normal displacements (as computed with the classical theory) could differ significantly from corresponding values of normal displacements computed with the use of the theories suggested in this book for certain values of E/E', G/G', h/a and h/b.

Tables below present values of the maximum normal displacement of a square plate at different values of E/E', G/G', for two different ratios of h/a (h/a = 1/5, 1/10). It was assumed for these calculations that $\nu = 0$ and $\nu' = 0.25$.

$h/a = 1/5$

E/E'	G/G'			
	0,0	1,0	2,5	5,0
0,0	1,000	1,158	1,395	1,789
1,0	0,980	1,138	1,375	1,770
5,0	0,901	1,059	1,296	1,691

$h/a = 1/10$

E/E'	G/G'			
	0,0	1,0	2,5	5,0
0,0	1,000	1,039	1,099	1,198
1,0	0,995	1,035	1,094	1,193
5,0	0,975	1,015	1,074	1,173

The corresponding value of $\bar{w}_0$, i.e., the value of the maximum deflection as obtained with the classical theory, was used as a multiplier in these tables [11]. The first lines in both tables show the results for computations carried out with equation (2.23), i.e., for the case when the effect of normal stress σ_z is disregarded.

It is easy to see from these results that the classical theory ($G/G' = 0$, $E/E' = 0$) produces significant errors for large values of G/G', even for a comparatively thin plate ($h/a = 1/5$).

The error in the classical theory decreases with the decrease in the relative plate thickness h^*, which for the case of the transversely isotropic plate is

$$h^* = \frac{h^2}{a^2}\left[2\frac{G}{G'} - \nu'\frac{E}{E'}\right]. \tag{2.24}$$

The above relative plate thickness depends in the general case on both the geometric and material characteristics of the plate.

By examining equations (2.5) – (2.9), and (2.18) – (2.23) and the tables for normal displacements, it is evident that the correction introduced into the classical theory by the normal stress σ_z is insignificant, as a rule, compared to the correction introduced due to the consideration of transverse shear [12, 13]. For example, when $h/a = 1/5$, $\nu = 0.3$, this correction amounts to 3%, while the correction introduced by the consideration of transverse shear reaches 22%.

The system of equations (2.11) – (2.13) can be reduced to two independent equations after simple manipulations:

$$D\Delta\Delta w = Z - \left(2\frac{G}{G'} - \nu'\frac{E}{E'}\right)\frac{h^2}{10(1-\nu)}\Delta Z, \tag{2.25}$$

$$\Delta\Phi - \frac{10}{h^2}\frac{G'}{G}\Phi = 0, \tag{2.26}$$

where $\Phi = \Phi(x, y)$ is a new undetermined function from which the functions φ, and ψ can be obtained:

$$\varphi = -\frac{E}{(1-\nu^2)}\frac{\partial}{\partial x}\Delta w - \frac{6}{5h(1-\nu)}\left(2\frac{G}{G'} - \nu'\frac{E}{E'}\right)\frac{\partial Z}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{12}{h^3} \tag{2.27}$$

$$\psi = -\frac{E}{(1-\nu^2)}\frac{\partial}{\partial y}\Delta w - \frac{6}{5h(1-\nu)}\left(2\frac{G}{G'} - \nu'\frac{E}{E'}\right)\frac{\partial Z}{\partial y} - \frac{\partial\Phi}{\partial x}\frac{12}{h^3}. \tag{2.28}$$

With expressions (2.27) and (2.28) it is easy to derive values for internal moment resultants and stresses, as well as the boundary conditions, using the usual initial equations.

In the particular case of an isotropic plate equations (2.14) and (2.15) are

$$D\Delta\Delta w = Z - \frac{h^2}{10}\frac{2-\nu}{1-\nu}\Delta Z,$$

$$\Delta\Phi - \frac{10}{h^2}\Phi = 0, \quad D = \frac{Eh^3}{12(1-\nu)},$$

These expressions are identical to those obtained by different authors and by different methods [14–17].

2. Thick plates. The theories suggested here for certain cases of boundary conditions and loads can be considered as approximate. They are, however, suitable for thick plates. To illustrate this point the maximum deflection of an isotropic plate simply-supported along the entire square contour (a x a) (h/a = 1/3, ν = 0.3) and loaded by the sinusoidal load (2.1) is given. All computations were carried out with equations (2.21), (2.23) and with the use of the precise theory of reference [18]. The results obtained are:

	Precise Theory	From (2.21)	From (2.23)	Classical Theory
$w_0 \frac{E}{qh}$	3.492	3.498	3.692	2.270

Looking at this table, it is easy to see that even in the case of such a thick plate (h/a = 1/3) the theories applied here result in insignificant errors. So, the formula (2.21) gives an error of 0.18%, the formula (2.23) – 5.73%. As for the classical theory of thin plates here, as would be expected, the error approaches 35%.

3. The function f(z). In Section 4 of Chapter II it was indicated that sufficient accuracy for the function $f_i(z)$ can be obtained with expression (2.4.1), namely:

$$f(z) = f_1(z) = f_2(z) = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right), \tag{2.29}$$

and therefore

$$\left.\begin{aligned} J_0 = J_{01} = J_{02} = \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right), \quad J_1 = J_2 = 0, \\ J_3 = J_4 = \frac{h^5}{120}, \quad J_5 = J_6 = K_i^- = \frac{h^3}{12}, \quad K_i^+ = 0. \end{aligned}\right\} \tag{2.30}$$

However, as was indicated before, such an expression for the function f(z), in the general case, only approximately represents the real picture.

To what extent then will the results change if $f_i(z)$ has other distributions? In order to answer this question, one additional distribution of $f_i(z)$ will be considered.

Let

$$f'(z) = f_1'(z) = f_2'(z) = \frac{1}{2}\left(\frac{h^4}{16} - z^4\right); \tag{2.31}$$

then

$$\left.\begin{aligned} J_0' = J_{10}' = J_{20}' = \frac{z}{2}\left(\frac{h^4}{16} - \frac{z^4}{5}\right), \quad J_1' = J_2' = 0, \\ J_3' = J_4' = \frac{h^7}{420}, \quad J_5' = J_6' = K_i^{-\prime} = \frac{h^5}{40}, \quad K_i^{+\prime} = 0. \end{aligned}\right\} \tag{2.32}$$

In order to compare results for both distributions of f(z) let us return to the square transversely isotropic plate which is simply-supported along the entire contour and loaded with the loads (2.1). To simplify the calculations the case is considered when the effect of σ_z can be disregarded ($A_1 = A_2 = 0$).

From equations (2.3.3) – (2.3.5) with the use of (2.10) we have for the transversely isotropic plate

$$\left.\begin{aligned} \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y} = -\frac{q}{J_5}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \\ D\frac{\partial}{\partial x}\Delta w - \frac{EJ_3}{G'(1-\nu^2)}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2\varphi}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2\psi}{\partial x\,\partial y}\right) + J_5\varphi = 0, \\ D\frac{\partial}{\partial y}\Delta w - \frac{EJ_4}{G'(1-\nu^2)}\left(\frac{\partial^2\psi}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2\psi}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2\varphi}{\partial x\,\partial y}\right) + J_6\psi = 0. \end{aligned}\right\} \tag{2.33}$$

Proceeding as before, the boundary conditions (2.4) are satisfied with (2.3), and undetermined coefficients A, B and C are found from (2.33); when their values are substituted into (2.3) the expressions for the functions w, φ, and ψ are obtained:

$$w = w_0\left[1 + \frac{4\pi^2}{(1-\nu)a^2}\frac{G}{G'}\frac{J_3}{J_5}\right], \tag{2.34}$$

$$\varphi = \frac{qa}{2\pi J_5}\cos\frac{\pi x}{a}\sin\frac{\pi y}{a}, \tag{2.35}$$

$$\psi = \frac{qa}{2\pi J_5}\sin\frac{\pi x}{a}\cos\frac{\pi y}{a}, \tag{2.36}$$

where for w_0

$$w_0 = \frac{3(1-\nu^2)a^4 q}{\pi^4 E h^3}\sin\frac{\pi x}{a}\sin\frac{\pi y}{a}.$$

Substituting values of J_i from (2.30) and (2.32) into equation (2.34) we obtain solutions for the normal displacements $w^{(i)}$ (both distributions of f(z))

$$w = w_0\left(1 + \frac{1}{5}h^*\right), \quad w' = w_0\left(1 + \frac{4}{21}h^*\right), \tag{2.37}$$

where the relative thickness h* is given as

$$h^* = \frac{2\pi^2 h^2}{(1-\nu)a^2}\frac{G}{G'}.$$

The ratio $w^{(i)}/w_0$ for different values of h* are tabulated below:

h*	0.5	1.0	2.0	5.0	10.0
w/w_0	1.100	1.200	1.400	2.000	3.000
w'/w_0	1.095	1.191	1.381	1.952	2.905

It is evident from this table that both distributions produce almost identical values of deflections. Even in cases when values of deflections from the classical theory differ from those obtained with the theory suggested in this book by 200 - 300%, the distributions of f(z) produce results for deflections which differ by only 2.5 – 3.0%.

The same conclusion is valid for other pertinent quantities of the plate. For example, substituting values of J_5 from (2.30) and (2.32) into (2.35) and (2.36) leads to expressions for the functions $\varphi^{(i)}$ and $\psi^{(i)}$:

$$\varphi = \frac{6qa}{\pi h^3}\cos\frac{\pi x}{a}\sin\frac{\pi y}{a}, \quad \psi = \frac{6qa}{\pi h^3}\sin\frac{\pi x}{a}\cos\frac{\pi y}{a}, \tag{2.38}$$

$$\varphi' = \frac{20qa}{\pi h^5}\cos\frac{\pi x}{a}\sin\frac{\pi y}{a}, \quad \psi' = \frac{20qa}{\pi h^5}\sin\frac{\pi x}{a}\cos\frac{\pi y}{a}. \tag{2.39}$$

Examination of $\varphi^{(i)}$ (analogously of $\psi^{(i)}$) reveals that in each case their values differ. This is to be expected because the $\varphi^{(i)}$ for these different cases represent different things and they cannot be compared. (This is obvious when the equations for τ_{xz} and τ_{yz} are compared, or the dimension of the corresponding variations of $\varphi^{(i)}$, are checked.) However, a substitution of values of $w^{(i)}$, $\varphi^{(i)}$ and $\psi^{(i)}$ into the corresponding equations for displacements (2.2.18), (2.2.19) or stresses (2.2.30) – (2.2.32) indicates that there is complete agreement between values of the two considered distributions. In any case displacements and stresses for these two cases differ by not more than normal displacements [19]. For example, substituting values of $\varphi^{(i)}$ and $\psi^{(i)}$ from (2.38) and (2.39) into (2.2.18) and assuming u = 0, the following inplane displacements are obtained for the surface points (z = h/2) of the transversely isotropic plate:

$$u_x = \left[-\frac{h}{2}\frac{\pi}{a}\bar{w}\left(1+\frac{1}{5}h^*\right) + \frac{qa}{4\pi G'}\right]\cos\frac{\pi x}{a}\sin\frac{\pi y}{a},$$

$$u_x' = \left[-\frac{h}{2}\frac{\pi}{a}\bar{w}\left(1+\frac{4}{21}h^*\right) + \frac{qa}{4\pi G'}\right]\cos\frac{\pi x}{a}\sin\frac{\pi y}{b},$$

This proves once more the accuracy of our statement.

It should be said in conclusion that an arbitrary selection of the functions f(z) in the suggested theory does not introduce inadmissible errors into the "internal" problem of the transverse bending of the plate.

4. Particular Theory. The problem of bending of a simply-supported rectangular plate can also be solved with the use of the particular theory (Chapter II, Section 6). Only the case of the transversely isotropic plate is considered.

From equalities (2.6.29), (2.10) and (2.6.28) the following governing differential equation is obtained

$$D\Delta\Delta w = Z - \left(2\frac{G}{G'} - \nu'\frac{E}{E'}\right)\frac{h^2}{10(1-\nu)}\Delta Z, \tag{2.40}$$

where, as before

$$Z = Z_2 = q\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{2.41}$$

The governing equation of the problem obtained in such a way is identical, obviously, with the corresponding equation of the general theory, i.e., with equation (2.25). However, as was expected, the second governing equation of the general theory, namely, (2.26), is absent here.

The boundary conditions of simple supports (2.6.18) are in this case

at x = 0, x = a;

$$w = 0, \quad D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) - \frac{h^5}{60(1-\nu)}\frac{G}{G'}\left(\frac{\partial\varphi_0}{\partial x} + \nu\frac{\partial\psi_0}{\partial y}\right) - \frac{\nu' E}{E'(1-\nu)}\frac{h^2}{10}Z = 0;$$

at y = 0, y = b;

$$w = 0, \quad D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) - \frac{h^5}{60(1-\nu)}\frac{G}{G'}\left(\frac{\partial\psi_0}{\partial y} + \nu\frac{\partial\varphi_0}{\partial x}\right) - \frac{\nu' E}{E'(1-\nu)}\frac{h^2}{10}Z = 0.$$

For this problem the boundary conditions can be simplified. We consider that φ_0 and ψ_0, on the basis of (2.6.31), (2.6.32) and (2.10), are of the following form:

$$\varphi_0 = -\frac{E}{1-\nu^2}\left(\frac{\partial^3 w_0}{\partial x^3} + \frac{\partial^3 w_0}{\partial x\,\partial y^2}\right), \quad \psi_0 = -\frac{E}{1-\nu^2}\left(\frac{\partial^3 w_0}{\partial y^3} + \frac{\partial^3 w_0}{\partial y\,\partial x^2}\right), \tag{2.42}$$

where w_0 is the solution of the classical equation

$$\Delta\Delta w_0 = \frac{q}{D}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

with simply-supported boundaries. This solution is [20]

$$w_0 = \frac{12(1-\nu^2)a^4b^4q}{\pi^4 Eh^3(a^2+b^2)^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \tag{2.43}$$

Using (2.42) and (2.43) the boundary conditions of the problem are

at x = 0, x = a;

$$w = 0, \quad \frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} = 0;$$

at y = 0, y = b;

$$w = 0, \quad \frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2} = 0. \tag{2.44}$$

Thus the solution for the transverse bending of a rectangular plate simply-supported along the entire contour and loaded by a sinusoidal load is reduced in the particular theory to the solution of the differential equation (2.40) with the boundary conditions (2.44).

Assuming that

$$w = A\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \tag{2.45}$$

the boundary conditions (2.44) are satisfied, and from the differential equation (2.40) an expression for the undetermined coefficient A is obtained:

$$A = \frac{12(1-\nu^2)a^4b^4q}{\pi^4 Eh^3(a^2+b^2)^2}\left[1+\frac{\pi^2h^2}{10(1-\nu)}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(2\frac{G}{G'}-\nu'\frac{E}{E'}\right)\right]. \tag{2.46}$$

Substituting the value of A from (2.46) into (2.45), a final equation for the normal displacement w is obtained. Inspection of this equation shows that it is identical to that obtained from the general theory for w, i.e., with formula (2.18). The same is true for other quantities (for example, φ_0 and φ; ψ_0 and ψ).

However, a comparison of the governing equations of the general and particular theories, as well as the analysis of problems solved in Section 1 of this Chapter, show that solutions obtained from the general and particular theories are not identical. In some cases the particular theory can produce a complete solution of the problem which would be identical to the solution obtained with the general theory. One such problem is this transverse bending of the plate. In particular, in the case of a transversely isotropic plate the equality of results found from both theories is verified by equal satisfaction of equation (2.26).

3. Bending of Simply-Supported Orthotropic Rectangular Plate by Arbitrary Load

A solution for the transverse bending of a simply-supported rectangular plate by arbitrary normally applied loads (X = 0, Y = 0) can be obtained with the use of the results from the preceding section.

We consider a rectangular orthotropic plate with principal direction of elasticity which coincides with the plate edges (Figure 13), simply-supported at all four sides (x = 0, x = a, y = 0, and y = b) and bent by a load distributed according to some arbitrary law:

$$Z_1 = 0, \qquad Z_2 = Z = F(x, y). \tag{3.1}$$

The system of differential equations for this problem will be of the type (2.5.5) – (2.5.7) and the applicable boundary conditions are given in (2.4).

The boundary conditions (2.4) will be satisfied if the solution of the system (2.5.5) – (2.5.7) is represented in the form of the following trigonometric series

$$\left.\begin{aligned} w &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \\ \varphi &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{\varphi mn} \cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \\ \psi &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{\psi mn} \sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}. \end{aligned}\right\} \tag{3.2}$$

In order to determine the unknown coefficients f_{mn}, $C_{\varphi mn}$, $C_{\psi mn}$, the load function F(x, y) is expanded into a double Fourier series

$$Z = F(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{3.3}$$

where

$$a_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy. \tag{3.4}$$

Now the expressions for w, φ, ψ, and Z from (3.2) and (3.3) are substituted into the system of equations (2.5.5) – (2.5.7), respectively, with the assumption that the function F(x, y) can be differentiated once with respect to x or to y. Comparing coefficients of identical trigonometric functions on the left- and right-hand sides of the resulting equations, we obtain the coefficients f_{mn}, $C_{\varphi mn}$, and $C_{\psi mn}$ in the following form:

$$f_{mn} = \frac{\Delta_{1mn}}{\Delta_{0mn}} a_{mn}, \qquad C_{\varphi mn} = \frac{\Delta_{2mn}}{\Delta_{0mn}} a_{mn}, \qquad C_{\psi mn} = \frac{\Delta_{3mn}}{\Delta_{0mn}} a_{mn}, \tag{3.5}$$

where the Δ_{imn} are given as follows:

$$\Delta_{0mn} = \frac{h^3}{12}\left[D_{11}\frac{\pi^4 m^4}{a^4} + 2(D_{12} + 2D_{66})\frac{\pi^4 m^2 n^2}{a^2 b^2} + D_{22}\frac{\pi^4 n^4}{b^4}\right] + \\ + \frac{h^2}{10}\left[\left(D_{11}\frac{\pi^2 m^2}{a^2} + D_{66}\frac{\pi^2 n^2}{b^2}\right)\left(D_{22}\frac{\pi^2 n^2}{b^2} + D_{66}\frac{\pi^2 m^2}{a^2}\right) - \right. \\ \left. - (D_{12} + D_{66})^2 \frac{\pi^4 m^2 n^2}{a^2 b^2}\right]\left(a_{44}\frac{\pi^2 m^2}{a^2} + a_{55}\frac{\pi^2 n^2}{b^2}\right), \tag{3.6}$$

$$\Delta_{1mn} = \frac{12}{h^3}\left\{\frac{h^6}{144} + \frac{h^5}{120}\left[a_{55}\left(D_{11}\frac{\pi^2 m^2}{a^2} + D_{66}\frac{\pi^2 n^2}{b^2}\right) + \right.\right. \\ \left. + a_{44}\left(D_{22}\frac{\pi^2 n^2}{b^2} + D_{66}\frac{\pi^2 m^2}{a^2}\right)\right] - \\ - \frac{h^4}{100} a_{44} a_{55}\left[(D_{12} + D_{66})\frac{\pi^4 m^2 n^2}{a^2 b^2} - \left(D_{11}\frac{\pi^2 m^2}{a^2} + D_{66}\frac{\pi^2 n^2}{b^2}\right) \times \right. \\ \left.\left. \times \left(D_{22}\frac{\pi^2 n^2}{b^2} + D_{66}\frac{\pi^2 m^2}{a^2}\right)\right]\right\} + \frac{h^2}{10}\left\{A_1\left[\frac{h^3}{12}\frac{\pi^2 m^2}{a^2} + \right.\right. \\ + a_{44}\frac{h^2}{10}\frac{\pi^2 m^2}{a^2}\left(D_{22}\frac{\pi^2 n^2}{b^2} + D_{66}\frac{\pi^2 m^2}{a^2}\right) - a_{55}\frac{h^2}{10}(D_{12} + D_{66})\frac{\pi^4 m^2 n^2}{a^2 b^2}\Big] + \\ + A_2\left[\frac{h^3}{12}\frac{\pi^2 n^2}{b^2} + a_{55}\frac{h^2}{10}\frac{\pi^2 n^2}{b^2}\left(D_{11}\frac{\pi^2 m^2}{a^2} + D_{66}\frac{\pi^2 n^2}{b^2}\right) - \right. \\ \left.\left. - a_{44}\frac{h^2}{10}(D_{12} + D_{66})\frac{\pi^4 m^2 n^2}{a^2 b^2}\right]\right\}, \tag{3.7}$$

$$\Delta_{2mn} = \frac{12}{h^3}\frac{\pi m}{a}\left\{\frac{h^3}{12}\left[D_{11}\frac{\pi^2 m^2}{a^2} + (D_{12}+2D_{66})\frac{\pi^2 n^2}{b^2}\right] + \right.$$
$$+ \frac{h^2}{10}a_{44}\left[\left(D_{11}\frac{\pi^2 m^2}{a^2} + D_{66}\frac{\pi^2 n^2}{b^2}\right)\left(D_{22}\frac{\pi^2 n^2}{b^2} + D_{66}\frac{\pi^2 m^2}{a^2}\right) - \right.$$
$$\left.\left. - (D_{12}+D_{66})^2\frac{\pi^4 m^2 n^2}{a^2 b^2}\right]\right\} - \frac{h^2}{10}\frac{\pi^3}{ab^2}\left\{A_1\left[(D_{12}+2D_{66})\frac{\pi^2 m^2}{a^2} + \right.\right.$$
$$\left.\left. + D_{22}\frac{\pi^2 n^2}{b^2}\right] - A_2\left[(D_{12}+2D_{66})\frac{\pi^2 n^2}{b^2} + D_{11}\frac{\pi^2 m^2}{a^2}\right]\right\}, \qquad (3.8)$$

$$\Delta_{3mn} = \frac{12}{h^3}\frac{\pi n}{b}\left\{\frac{h^3}{12}\left[D_{22}\frac{\pi^2 n^2}{b^2} + (D_{12}+2D_{66})\frac{\pi^2 m^2}{a^2}\right] + \right.$$
$$+ \frac{h^2}{10}a_{55}\left[\left(D_{11}\frac{\pi^2 m^2}{a^2} + D_{66}\frac{\pi^2 n^2}{b^2}\right)\left(D_{22}\frac{\pi^2 n^2}{b^2} + D_{66}\frac{\pi^2 m^2}{a^2}\right) - \right.$$
$$\left.\left. - (D_{12}+D_{66})^2\frac{\pi^4 m^2 n^2}{a^2 b^2}\right]\right\} - \frac{h^2}{10}\frac{\pi^3 nm^2}{ba^2}\left\{A_2\left[(D_{12}+2D_{66})\frac{\pi^2 n^2}{b^2} + \right.\right.$$
$$\left.\left. + D_{11}\frac{\pi^2 m^2}{a^2}\right] - A_1\left[(D_{12}+2D_{66})\frac{\pi^2 m^2}{a^2} + D_{22}\frac{\pi^2 b^2}{b^2}\right]\right\}. \qquad (3.9)$$

Substituting the resulting values of the functions w, φ, and ψ into (2.5.11) – (2.5.17) we obtain the moment resultants and stresses:

$$M_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left(D_{11}\frac{\pi^2 m^2}{a^2} + D_{12}\frac{\pi^2 n^2}{b^2}\right)\Delta_{1mn} - \right.$$
$$\left. - \frac{h^2}{10}\left(a_{55}D_{11}\Delta_{2mn}\frac{\pi m}{a} + a_{44}D_{12}\Delta_{3mn}\frac{\pi n}{b} + A_1\Delta_{0mn}\right)\right] \times$$
$$\times \frac{a_{mn}}{\Delta_{0mn}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

$$M_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left(D_{22}\frac{\pi^2 n^2}{b^2} + D_{12}\frac{\pi^2 m^2}{a^2}\right)\Delta_{1mn} - \right.$$
$$\left. - \frac{h^2}{10}\left(a_{44}D_{22}\Delta_{3mn}\frac{\pi n}{b} + a_{55}D_{12}\Delta_{2mn}\frac{\pi m}{a} + A_2\Delta_{0mn}\right)\right] \times$$
$$\times \frac{a_{mn}}{\Delta_{0mn}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

$$H = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[-2D_{66}\Delta_{1mn}\frac{\pi^2 mn}{ab} + \\ + \frac{h^2}{10}D_{66}\Big(a_{55}\Delta_{2mn}\frac{\pi n}{b} + a_{44}\Delta_{3mn}\frac{\pi m}{a}\Big)\Big]\frac{a_{mn}}{\Delta_{0mn}}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b},$$

$$N_x = \frac{h^3}{12}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Delta_{2mn}}{\Delta_{0mn}}a_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

$$N_y = \frac{h^3}{12}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Delta_{3mn}}{\Delta_{0mn}}a_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b},$$

$$\sigma_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[z\Big(B_{11}\frac{\pi^2 m^2}{a^2} + B_{12}\frac{\pi^2 n^2}{b^2}\Big)\Delta_{1mn} - \\ - \frac{z}{2}\Big(\frac{h^2}{4} - \frac{z^2}{3}\Big)\Big(a_{55}B_{11}\Delta_{2mn}\frac{\pi m}{a} + a_{44}B_{12}\Delta_{3mn}\frac{\pi n}{b}\Big) - \\ - \frac{z}{h}\Big(\frac{3}{2} - 2\frac{z^2}{h^2}\Big)A_1\Delta_{0mn}\Big]\frac{a_{mn}}{\Delta_{0mn}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

$$\sigma_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[z\Big(B_{22}\frac{\pi^2 n^2}{b^2} + B_{12}\frac{\pi^2 m^2}{a^2}\Big)\Delta_{1mn} - \\ - \frac{z}{2}\Big(\frac{h^2}{4} - \frac{z^2}{3}\Big)\Big(a_{44}B_{22}\Delta_{3mn}\frac{\pi n}{b} + a_{55}B_{12}\Delta_{2mn}\frac{\pi m}{a}\Big) - \\ - \frac{z}{h}\Big(\frac{3}{2} - 2\frac{z^2}{h^2}\Big)A_2\Delta_{0mn}\Big]\frac{a_{mn}}{\Delta_{0mn}}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

$$\tau_{xy} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big[-2zB_{66}\Delta_{1mn}\frac{\pi^2 mn}{ab} + \frac{z}{2}\Big(\frac{h^2}{4} - \\ - \frac{z^2}{3}\Big)\Big(a_{55}\Delta_{2mn}\frac{\pi n}{b} + a_{44}\Delta_{3mn}\frac{\pi m}{a}\Big)\Big]\frac{a_{mn}}{\Delta_{0mn}}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}.$$

The coefficients a_{mn} present in the above expressions depend on the distribution of the external load F(x, y). For the case of a uniformly distributed load

$$a_{mn} = \frac{16q}{\pi^2}\frac{1}{mn} \qquad m, n = 1, 3, 5\ldots$$

$a_{mn} = 0$ for all remaining m and n; q is the load intensity.

For the case of a concentrated force P applied at the point (ξ,η)

$$a_{mn} = \frac{4P}{ab}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b} \qquad (m, n = 1, 2, 3, \ldots).$$

The results obtained here can be applied to other problems concerning simply-supported rectangular plates subjected to various types of loads [21].

4. Bending of Rectangular Plate with Two Supported Sides

It is assumed that the load and boundary conditions are such that $u = v = 0$. Furthermore, the plate is made of transversely isotropic material and the effect of the normal stress σ_z on the state of stress and normal displacement w of the plate are disregarded. A general solution of the transverse bending of the plate with an arbitrary transverse load is presented on the assumption that the plane of isotropy at each point of the plate is parallel to the plate middle plane. Two opposite sides of the plate are simply-supported and the other two are fixed arbitrarily [22].

The coordinate system is selected in such a way that the origin of coordinates is located at the center of a simply-supported side and the y-axis is directed along this same side, while the x-axis is perpendicular to it (Figure 14).

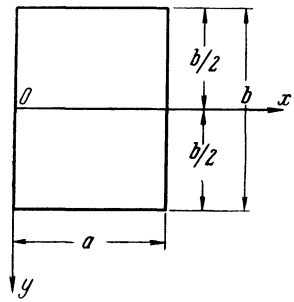

Figure 14.

The system of differential equations of bending (2.5.25) – (2.5.27) because of (2.10) are

$$\left.\begin{aligned} &\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y} = -\frac{12}{h^3}Z, \\ &D\frac{\partial}{\partial x}\Delta w - \frac{h^2 D}{10G'}\left(\frac{\partial^2\varphi}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2\varphi}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2\psi}{\partial x\,\partial y}\right) + \frac{h^3}{12}\varphi = 0, \\ &D\frac{\partial}{\partial y}\Delta w - \frac{h^2 D}{10G'}\left(\frac{\partial^2\psi}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2\psi}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2\varphi}{\partial x\,\partial y}\right) + \frac{h^3}{12}\psi = 0. \end{aligned}\right\} \tag{4.1}$$

The normal stress σ_z is disregarded here as compared with other stresses, i.e., it has been assumed that $A_1 = A_2 = 0$; and $Z_1 = 0$, $Z_2 = Z = Z(x, y)$.

From (2.5.18) – (2.5.24) we have

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) + \frac{h^2 D}{10G'}\left(\frac{\partial \varphi}{\partial x} + \nu\frac{\partial \psi}{\partial y}\right), \tag{4.2}$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) + \frac{h^2 D}{10G'}\left(\frac{\partial \psi}{\partial y} + \nu\frac{\partial \varphi}{\partial x}\right), \tag{4.3}$$

$$H = -(1-\nu)D\frac{\partial^2 w}{\partial x\,\partial y} + \frac{h^2(1-\nu)D}{20G'}\left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x}\right), \tag{4.4}$$

$$N_x = \frac{h^3}{12}\varphi, \qquad N_y = \frac{h^3}{12}\psi, \tag{4.5}$$

$$\sigma_x = -z\bar{B}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\frac{\bar{B}}{G'}\left(\frac{\partial \varphi}{\partial x} + \nu\frac{\partial \psi}{\partial y}\right), \tag{4.6}$$

$$\sigma_y = -z\bar{B}\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right) + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\frac{\bar{B}}{G'}\left(\frac{\partial \psi}{\partial y} + \nu\frac{\partial \varphi}{\partial x}\right), \tag{4.7}$$

$$\tau_{xy} = -2zG\frac{\partial^2 w}{\partial x\,\partial y} + \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\frac{G}{G'}\left(\frac{\partial \varphi}{\partial y} + \frac{\partial \psi}{\partial x}\right). \tag{4.8}$$

Since the plate is simply-supported along two opposite sides (x = 0, x = a), then from (2.3.14) for the simply-supported edges:

at x = 0, x = a

$$w = 0, \qquad M_x = 0, \qquad \psi = 0. \tag{4.9}$$

The solution of the system (4.1) will be determined in such a form that the boundary conditions (4.9) are satisfied completely. Namely, it is assumed that

$$\left.\begin{aligned} w &= \sum_{n=1}^{\infty} W_n(y)\sin\frac{n\pi x}{a}, \\ \varphi &= \sum_{n=1}^{\infty} \Phi_n(y)\cos\frac{n\pi x}{a}, \\ \psi &= \sum_{n=1}^{\infty} \Psi_n(y)\sin\frac{n\pi x}{a} \end{aligned}\right\} \tag{4.10}$$

Substituting values of w, φ and ψ from (4.10) into (4.1) and considering the external load in the form of the series

$$Z(x, y) = \sum_{n=1}^{\infty} P_n(y) \sin \frac{n\pi x}{a}, \tag{4.11}$$

where

$$P_n(y) = \frac{2}{a} \int_0^a Z(x, y) \sin \frac{n\pi x}{a}\, dx, \tag{4.11'}$$

the following system of ordinary differential equations with respect to the functions W_n, Φ_n and Ψ_n is obtained

$$\left.\begin{aligned}
&\lambda_n \Phi_n - \Psi_n' = \frac{12}{h^3} P_n(y), \\
&\lambda_n W_n'' - \lambda_n^3 W_n - \frac{(1-\nu)h^2}{20G'} \Phi_n'' + \\
&\quad + \left(\frac{1-\nu^2}{E} + \frac{h^2\lambda_n^2}{10G'}\right)\Phi_n - \frac{(1+\nu)h^2\lambda_n}{20G'} \Psi_n = 0, \\
&W_n''' - \lambda_n^2 W_n' + \frac{(1+\nu)h^2}{20G'} \Phi_n' - \frac{h^2}{10G'} \Psi_n'' + \\
&\quad + \left[\frac{1-\nu^2}{E} + \frac{(1-\nu)h^2\lambda_n^2}{20G'}\right] \Psi_n = 0,
\end{aligned}\right\} \tag{4.12}$$

where

$$\lambda_n = \frac{n\pi}{a}, \qquad (\ )' = \frac{d(\ )}{dy}. \tag{4.13}$$

The solution of a uniform system of equations such as (4.12) is sought in the form:

$$W_n = A_n e^{sy}, \qquad \Phi_n = B_n e^{sy}, \qquad \Psi_n = C_n e^{sy}. \tag{4.14}$$

Substituting (4.14) into (4.12) and carrying out several manipulations, the following characteristic equation is obtained:

$$(s^2 - \lambda_n^2)^2 \left[\frac{h^2(1-\nu)}{20G'}(s^2 - \lambda_n^2) - \frac{1-\nu^2}{E}\right] = 0, \tag{4.15}$$

the roots of which are

$$\left.\begin{aligned}
&s_{1,2} = \lambda_n, \qquad s_{3,4} = -\lambda_n, \\
&s_{5,6} = \pm\omega_n = \pm\sqrt{\lambda_n^2 + \frac{20G'(1+\nu)}{Eh^2}}.
\end{aligned}\right\} \tag{4.16}$$

On the basis of (4.14) and (4.16) the solution of the system of equations (4.12) is

$$W_n(y) = (A_{n1} + A_{n2}y)\,e^{\lambda_n y} + (A_{n3} + A_{n4}y)\,e^{-\lambda_n y} + \\ + A_{n5}e^{\omega_n y} + A_{n6}e^{-\omega_n y}, \quad (4.17)$$

$$\Phi_n(y) = (B_{n1} + B_{n2}\,y)\,e^{\lambda_n y} + (B_{n3} + B_{n4}\,y)\,e^{-\lambda_n y} + \\ + B_{n5}\,e^{\omega_n y} + B_{n6}\,e^{-\omega_n y}, \quad (4.18)$$

$$\Psi_n(y) = (C_{n1} + C_{n2}\,y)\,e^{\lambda_n y} + (C_{n3} + C_{n4}\,y)\,e^{-\lambda_n y} + \\ + C_{n5}\,e^{\omega_n y} + C_{n6}\,e^{-\omega_n y}, \quad (4.19)$$

where A_{ni}, B_{ni} and C_{ni} are eighteen intergration constants which are related by the following identities because of corresponding equations of the system (4.12):

$$\left.\begin{aligned} B_{n1} = C_{n1} = -\frac{2E\lambda_n^2}{1-\nu^2}A_{n2}, \qquad & B_{n5} = \frac{\omega_n}{\lambda_n}C_{n5}, \\ B_{n3} = -C_{n3} = \frac{2E\lambda_n^2}{1-\nu^2}A_{n4}, \qquad & B_{n6} = -\frac{\omega_n}{\lambda_n}C_{n6}, \\ B_{n2} = C_{n2} = 0, \qquad & A_{n5} = 0, \\ B_{n4} = C_{n4} = 0, \qquad & A_{n6} = 0; \end{aligned}\right\} \quad (4.20)$$

Due to these relationships, only six integration constants remain instead of 18. These six constants are determined from the boundary conditions along the edges y = ±b/2.

Considering the identities (4.20) for W_n, Φ_n and Ψ_n, (4.17) – (4.19) can be rewritten:

$$W_n = (A_{n1} + yA_{n2})\,e^{\lambda_n y} + (A_{n3} + yA_{n4})\,e^{-\lambda_n y},$$

$$\Phi_n = -\frac{2E\lambda_n^2}{1-\nu^2}A_{n2}e^{\lambda_n y} + \frac{2E\lambda_n^2}{1-\nu^2}A_{n4}e^{-\lambda_n y} + \\ + \frac{\omega_n}{\lambda_n}C_{n5}e^{\omega_n y} - \frac{\omega_n}{\lambda_n}C_{n6}e^{-\omega_n y},$$

$$\Psi_n = -\frac{2E\lambda_n^2}{1-\nu^2}A_{n2}e^{\lambda_n y} - \frac{2E\lambda_n^2}{1-\nu^2}\,A_{n4}e^{-\lambda_n y} + C_{n5}e^{\omega_n y} + C_{n6}e^{-\omega_n y}.$$

Using the same notation for all the integration constants (K_{ni}) and determining a particular solution of the non-uniform system (4.12) by the method of variation of parameters, the following expressions are obtained for these functions:

$$W_n(y) = K_{n1}\,\mathrm{sh}\,\lambda_n y + K_{n2}\,\mathrm{ch}\,\lambda_n y + K_{n3}y\,\mathrm{sh}\,\lambda_n y + \\ + K_{n4}y\,\mathrm{ch}\,\lambda_n y + W_n^*(y), \quad (4.21)$$

$$\Phi_n(y) = -\frac{24D\lambda_n^2}{h^3}(K_{n3}\operatorname{ch}\lambda_n y + K_{n4}\operatorname{sh}\lambda_n y) + \\ + \frac{\omega_n}{\lambda_n}(K_{n5}\operatorname{sh}\omega_n y + K_{n6}\operatorname{ch}\omega_n y) + \Phi_n^*(y), \quad (4.22)$$

$$\Psi_n(y) = -\frac{24D\lambda_n^2}{h^3}(K_{n3}\operatorname{sh}\lambda_n y + K_{n4}\operatorname{ch}\lambda_n y) + \\ + K_{n5}\operatorname{ch}\omega_n y + K_{n6}\operatorname{sh}\omega_n y + \Psi_n^*(y), \quad (4.23)$$

The particular solutions of the non-uniform system (4.12) are:

$$W_n^*(y) = -\frac{1}{4D\lambda_n^3}[(1 + 2k\lambda_n^2 - \lambda_n y)J_{n1}e^{\lambda_n y} - \\ -(1 + 2k\lambda_n^2 + \lambda_n y)J_{n2}e^{-\lambda_n y} + \lambda_n(J_{n3}e^{\lambda_n y} + J_{n4}e^{-\lambda_n y})], \quad (4.24)$$

$$\Phi_n^*(y) = -\frac{6}{h^3}(J_{n1}e^{\lambda_n y} - J_{n2}e^{-\lambda_n y}), \quad (4.25)$$

$$\Psi_n^*(y) = -\frac{6}{h^3}(J_{n1}e^{\lambda_n y} + J_{n2}e^{-\lambda_n y}). \quad (4.26)$$

where the following notation has been used:

$$\left.\begin{aligned} J_{n1} &= \int e^{-\lambda_n y}P_n(y)\,dy, \quad J_{n2} = \int e^{\lambda_n y}P_n(y)\,dy, \\ J_{n3} &= \int y e^{-\lambda_n y}P_n(y)\,dy, \quad J_{n4} = \int y e^{\lambda_n y}P_n(y)\,dy, \end{aligned}\right\} \quad (4.27)$$

$$k = \frac{Eh^2}{10(1-\nu^2)G'}, \qquad \omega_n = \sqrt{\lambda_n^2 + \frac{2}{(1-\nu)k}}. \quad (4.28)$$

Two problems on the bending of rectangular plates subjected to a uniformly distributed load of intensity Z(x, y) = q = constant are considered below. In this case using formulas (4.11′) and (4.24) – (4.26)

$$\left.\begin{aligned} P_n(y) &= \frac{4q}{\pi n}, \quad W_n^*(y) = \frac{4q(1 + k\lambda_n^2)}{\pi n D\lambda_n^4}, \\ \Phi_n^*(y) &= \frac{48q}{\pi n h^3\lambda_n}, \quad \Psi_n^*(y) = 0, \end{aligned}\right\} \quad (4.29)$$

Here n can have only odd values (n = 1, 3, 5, . . .).

1. Plate with entire contour simply-supported. Consider the problem of the transverse bending of a rectangular plate subjected to a uniformly distributed load q = constant with the entire contour simply-supported, i.e.,

along with boundary conditions (4.9) we also have

at $y = b/2$, $y = -b/2$

$$w = 0, \qquad M_y = 0, \qquad \varphi = 0. \tag{4.30}$$

The deformed surface of the plate in this case is symmetrical with respect to the x-axis (Figure 14) and, therefore, functions $\Phi_n(y)$ and $W_n(y)$ are even and $\Psi_n(y)$ is odd. Therefore, in expressions (4.21) – (4.23) for this problem it should be assumed that

$$K_{n1} = K_{n4} = K_{n5} = 0$$

Considering these equalities and also (4.29) the solution can be written:

$$\left.\begin{aligned} W_n(y) &= K_{n2} \operatorname{ch} \lambda_n y + K_{n3} y \operatorname{sh} \lambda_n y + \frac{4q(1 + k\lambda_n^2)}{\pi n D \lambda_n^4}, \\ \Phi_n(y) &= -\frac{24D\lambda_n^2}{h^3} K_{n3} \operatorname{ch} \lambda_n y + \frac{\omega_n}{\lambda_n} K_{n6} \operatorname{ch} \omega_n y + \frac{48q}{\pi h^3 n \lambda_n}, \\ \Psi_n(y) &= -\frac{24D\lambda_n^2}{h^3} K_{n3} \operatorname{sh} \lambda_n y + K_{n6} \operatorname{sh} \omega_n y. \end{aligned}\right\} \tag{4.31}$$

From equations (4.3) and (4.10) and the indicated symmetry of the deformed plate, the boundary conditions (4.30) can be reduced to

$$W_n\left(\frac{b}{2}\right) = 0, \qquad \Phi_n\left(\frac{b}{2}\right) = 0,$$

$$W_n''\left(\frac{b}{2}\right) + \frac{h^2}{10G'}\left[\nu\lambda_n \Phi_n\left(\frac{b}{2}\right) - \Psi_n'\left(\frac{b}{2}\right)\right] = 0.$$

Substituting the functions in the form (4.31) into these transformed boundary conditions, we obtain a system of three algebraic equations with respect to the three integration constants, K_{n2}, K_{n3}, K_{n6}. The solution of this system results in

$$\left.\begin{aligned} K_{n2} &= -\frac{qb}{\pi n D \lambda_n^3} \frac{\operatorname{sh} \alpha_n}{\operatorname{ch}^2 \alpha_n} - \frac{4q(1 + k\lambda_n^2)}{\pi n D \lambda_n^4} \frac{1}{\operatorname{ch} \alpha_n}, \\ K_{n3} &= \frac{2q}{\pi n D \lambda_n^3} \frac{1}{\operatorname{ch} \alpha_n}, \qquad K_{n6} = 0, \end{aligned}\right\} \tag{4.32}$$

where

$$\alpha_n = \frac{\lambda_n b}{2} = \frac{n\pi b}{2a}. \tag{4.33}$$

Finally, using equations (4.10), (4.31) and (4.32) the solution is:

$$w = \frac{4qa^4}{\pi^5 D} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^5} \left\{ \left[1 - \frac{\alpha_n \operatorname{th} \alpha_n + 2}{2 \operatorname{ch} \alpha_n} \operatorname{ch} \lambda_n y + \right.\right.$$
$$\left.\left. + \frac{\lambda_n}{2 \operatorname{ch} \alpha_n} y \operatorname{sh} \lambda_n y \right] + k\lambda_n^2 \left(1 - \frac{\operatorname{ch} \lambda_n y}{\operatorname{ch} \alpha_n} \right) \right\} \sin \lambda_n x, \tag{4.34}$$

$$\varphi = \frac{48qa}{\pi^2 h^3} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^2} \left(1 - \frac{\operatorname{ch} \lambda_n y}{\operatorname{ch} \alpha_n} \right) \cos \lambda_n x, \tag{4.35}$$

$$\psi = -\frac{48qa}{\pi^2 h^3} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^2} \frac{\operatorname{sh} \lambda_n y}{\operatorname{ch} \alpha_n} \sin \lambda_n x. \tag{4.36}$$

With the expressions for w, φ and ψ and equations (4.2) – (4.8) it is easy to find expressions for the stresses and moment resultants. For simplicity we give here only expressions for the derivatives which are necessary to evaluate these stresses and moment resultants, namely:

$$\frac{\partial^2 w}{\partial x^2} = -\frac{4qa^4}{\pi^5 D} \sum_{n=1,3,\ldots}^{\infty} \frac{\lambda_n^2}{n^5} \left\{ \left[1 - \frac{\alpha_n \operatorname{th} \alpha_n + 2}{2 \operatorname{ch} \alpha_n} \operatorname{ch} \lambda_n y + \right.\right.$$
$$\left.\left. + \frac{\lambda_n}{2 \operatorname{ch} \alpha_n} y \operatorname{sh} \lambda_n y \right] + k\lambda_n^2 \left(1 - \frac{\operatorname{ch} \lambda_n y}{\operatorname{ch} \alpha_n} \right) \right\} \sin \lambda_n x,$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{4qa^4}{\pi^5 D} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^5} \left\{ \left[1 - \frac{\alpha_n \operatorname{th} \alpha_n + 2}{2 \operatorname{ch} \alpha_n} \lambda_n^2 \operatorname{ch} \lambda_n y + \right.\right.$$
$$\left.\left. + \frac{\lambda_n^2}{\operatorname{ch} \alpha_n} \operatorname{ch} \lambda_n y + \frac{\lambda_n^3}{2 \operatorname{ch} \alpha_n} y \operatorname{sh} \lambda_n y \right] + k\lambda_n^2 \left(1 - \lambda_n^2 \frac{\operatorname{ch} \lambda_n y}{\operatorname{ch} \alpha_n} \right) \right\} \sin \lambda_n x,$$

$$\frac{\partial^2 w}{\partial x \, \partial y} = \frac{4qa^4}{\pi^5 D} \sum_{n=1,3,\ldots}^{\infty} \frac{\lambda_n}{n^5} \left\{ \left[1 - \frac{\alpha_n \operatorname{th} \alpha_n + 2}{2 \operatorname{ch} \alpha_n} \lambda_n \operatorname{sh} \lambda_n y + \right.\right.$$
$$\left.\left. + \frac{\lambda_n}{2 \operatorname{ch} \alpha_n} \operatorname{sh} \lambda_n y + \frac{\lambda_n^2}{2 \operatorname{ch} \alpha_n} y \operatorname{ch} \lambda_n y \right] + k\lambda_n^2 \left(1 - \lambda_n \frac{\operatorname{sh} \lambda_n y}{\operatorname{ch} \alpha_n} \right) \right\} \cos \lambda_n x,$$

$$\frac{\partial \varphi}{\partial x} = -\frac{48qa}{\pi^2 h^3} \sum_{n=1,3,\ldots}^{\infty} \frac{\lambda_n}{n^2} \left(1 - \frac{\operatorname{ch} \lambda_n y}{\operatorname{ch} \alpha_n} \right) \sin \lambda_n x,$$

$$\frac{\partial \varphi}{\partial y} = \frac{48qa}{\pi^2 h^3} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n^2}\left(1 - \lambda_n \frac{\operatorname{sh} \lambda_n y}{\operatorname{ch} \alpha_n}\right)\cos \lambda_n x,$$

$$\frac{\partial \psi}{\partial x} = -\frac{48qa}{\pi^2 h^3} \sum_{n=1,3,\ldots}^{\infty} \frac{\lambda_n}{n^2} \frac{\operatorname{sh} \lambda_n y}{\operatorname{ch} \alpha_n} \cos \lambda_n x,$$

$$\frac{\partial \psi}{\partial y} = -\frac{48qa}{\pi^2 h^3} \sum_{n=1,3,\ldots}^{\infty} \frac{\lambda_n}{n^2} \frac{\operatorname{ch} \lambda_n y}{\operatorname{ch} \alpha_n} \sin \lambda_n x.$$

In order to evaluate quantitatively the effect of transverse shear on the plate deflections, the maximum deflection at x = a/2, y = 0 is given below

$$w_{\max} = \frac{4qa^4}{\pi^5 D}\left\{ \sum_{n=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^5}\left[1 - \frac{\alpha_n \operatorname{th} \alpha_n + 2}{2 \operatorname{ch} \alpha_n}\right] + \right.$$

$$\left. + \frac{\pi^2}{a^2} k \sum_{n=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3}\left(1 - \frac{1}{\operatorname{ch} \alpha_n}\right)\right\}. \qquad (4.37)$$

The second summation in (4.37) contains the multiplier k which characterizes the effect of transverse shear and which is given in (4.28). Assuming that k = 0, an expression for the maximum deflection is obtained which is equivalent to that obtained with the classical theory [23].

As an example, let us compute the maximum deflection of a square plate (a = b). From (4.37) because of (4.28)

$$w_{\max} = \bar{w}_0\left[1 + \frac{1{,}8145 E h^2}{(1 - \nu^2) a^2 G'}\right], \qquad (4.38)$$

where $\bar{w}_0 = 0.00406\ qa^4/D$ is the maximum deflection of a square plate computed from the classical theory.

The table below presents the ratio of the maximum deflection of a square plate to the corresponding value as found from the classical theory for different values of the ratios E/G′ and h/a with $\nu = 0.3$.

h/a	E/G′			
	0.0	2.6	5.0	10.0
0.1	1.0	1.0518	1.0997	1.1994
0.2	1.0	1.2074	1.3988	1.7976

Here E/G′ = 2.6 corresponds to an isotropic plate.

2. Plate with two opposite edges rigidly-supported. Let a rectangular plate which is bent by a uniformly distributed load q = constant be rigidly-supported along the edges y = ±b/2 and simply-supported along the edges x = 0, x = a (Figure 14).

On the basis of equations (2.3.14), (2.3.15), (2.4.1) and (2.4.2) or (2.29), (2.30), the boundary conditions of the problem are:

at x = 0, x = a

$$w = 0, \qquad M_x = 0, \qquad \psi = 0; \tag{4.39}$$

at y = ±b/2

$$\left.\begin{aligned} w = 0, \qquad \varphi = 0, \\ -\frac{\partial w}{\partial y} + \frac{1}{2G'}\left(\frac{h^2}{4} - \frac{z_0^2}{3}\right)\psi = 0. \end{aligned}\right\} \tag{4.40}$$

Just as in the previous problem the deformed surface of the plate is symmetrical with respect to the x-axis. Consequently the functions $W_n(y)$, $\Phi_n(y)$ and $\Psi_n(y)$ can be expressed in the form (4.31).

From (4.3) and (4.10) and the indicated symmetry of the deformed surface of the plate, the boundary conditions (4.40) are reduced to

$$W_n\left(\frac{b}{2}\right) = 0, \qquad \Phi_n\left(\frac{b}{2}\right) = 0,$$

$$-W_n'\left(\frac{b}{2}\right) + \frac{1}{2G'}\left(\frac{h^2}{4} - \frac{z_0^2}{3}\right)\Psi_n\left(\frac{b}{2}\right) = 0.$$

Substituting the expressions for W_n, Φ_n and Ψ_n from (4.31) into these transformed boundary conditions, a system of three algebraic equations with respect to the three integration constants, K_{n2}, K_{n3}, K_{n6}, is obtained.

Upon solving this system we find that

$$K_{n2} = \frac{\Delta_{n2}}{\Delta_n}, \qquad K_{n3} = \frac{\Delta_{n3}}{\Delta_n}, \qquad K_{n6} = \frac{\Delta_{n6}}{\Delta_n}, \tag{4.41}$$

where

$$\Delta_n = \frac{\omega_n}{2}(2\alpha_n + \operatorname{sh} 2\alpha_n)\operatorname{ch}\beta_n +$$

$$+ 10k\lambda_n^2\left(\frac{1}{4} - \frac{z_0^2}{3h^2}\right)(\omega_n \operatorname{sh}\alpha_n \operatorname{ch}\beta_n - \lambda_n \operatorname{ch}\alpha_n \operatorname{sh}\beta_n)\operatorname{ch}\alpha_n,$$

$$\Delta_{n2} = \frac{4q}{\pi n D}\left\{-\frac{1 + k\lambda_n^2}{\lambda_n^4}\omega_n \operatorname{ch}\beta_n(\operatorname{sh}\alpha_n + \alpha_n \operatorname{ch}\alpha_n) + \right.$$

$$\left. + 10k\left(\frac{1}{4} - \frac{z_0^2}{3h^2}\right)\left[\frac{b}{4}\operatorname{sh}\alpha_n \operatorname{sh}\beta_n + \right.\right.$$

$$+\frac{1+k\lambda_n^2}{\lambda_n^2}(\lambda_n \operatorname{ch}\alpha_n \operatorname{sh}\beta_n - \omega_n \operatorname{sh}\alpha_n \operatorname{ch}\beta_n)\Big]\Big\},$$

$$\Delta_{n3} = \frac{4q}{\pi n D}\left[\frac{\omega_n}{\lambda_n^3}(1+k\lambda_n^2)\operatorname{sh}\alpha_n \operatorname{ch}\beta_n - \right.$$

$$\left. - \frac{6D}{hG'}\left(\frac{1}{4} - \frac{z_0^2}{3h^2}\right)\operatorname{ch}\alpha_n \operatorname{sh}\beta_n\right],$$

$$\Delta_{n6} = \frac{24q}{\pi n h^3}(\operatorname{sh} 2\alpha_n - 2\alpha_n) + 2k\lambda_n^2\left[1 - 5\left(\frac{1}{4} - \frac{z_0^2}{3h^2}\right)\right]\operatorname{sh} 2\alpha_n,$$

$$\alpha_n = \frac{\lambda_n b}{2}, \quad \beta_n = \frac{\omega_n b}{2}.$$

Thus, using formulas (4.31), (4.10) and the integration constants (4.41), the basic solution of the problem is determined. That is, the functions w, φ, and ψ and consequently all other quantities of interest can be determined. However, these formulas are not given here because of their size. On the basis of these results and passing to the limit ($k \to 0$) the corresponding formulas of the classical theory are obtained.

For a quantitative evaluation of the effect of transverse shear on the plate deflections, an expression for the maximum deflection at x = a/2, y = 0 will be given, as well as some numerical results:

$$w_{\max} = \sum_{n=1,3,\ldots}^{\infty} (-1)^{\frac{n-1}{2}}\left(\frac{4q}{\pi D}\frac{1+k\lambda_n^2}{n\lambda_n^4} + \frac{\Delta_{n2}}{\Delta_n}\right). \qquad (4.42)$$

Passing to the limit ($k \to 0$) we have the maximum plate deflection $\bar{w}_0$ as obtained with the classical theory:

$$\bar{w}_0 = \frac{4qa^4}{\pi^5 D}\sum_{n=1,3,\ldots}^{\infty}\frac{(-1)^{\frac{n-1}{2}}}{n^5}\left[1 - \frac{2(\operatorname{sh}\alpha_n + \alpha_n \operatorname{ch}\alpha_n)}{2\alpha_n + \operatorname{sh} 2\alpha_n}\right]. \qquad (4.43)$$

In the case of a square plate (b = a) from (4.43) using only the first three terms in the series we obtain

$$\bar{w}_0 = 0.001918\frac{qa^4}{D}.$$

Values of w_{max} / w_0 for a square plate at different ratios of E/G', h/a, and z_0/h (see section 2.3) with $\nu = 0.3$, are given in the table below.

z_0/h	h/a	E/G'			z_0/h	h/a	E/G'		
		0,0	2,6	5,0			0,0	2,6	5,0
0,5	0,1	1,0	1,0813	1,1569	0,1	0,1	1,0	1,1076	1,2057
0,5	0,2	1,0	1,3239	1,6284	0,1	0,2	1,0	1,4217	1,7929

A comparison of the numerical ratios of deflections for a square plate with two fixed and two simply-supported edges with the results obtained for a plate simply-supported along the entire contour (i.e., comparing the last two tables) shows that the boundary conditions influence considerably the values of the plate deflection when including transverse shear.

The method of analyzing the transverse bending of a plate with two simply-supported edges is also completely applicable to an orthotropic plate. However, in this case the characteristic equation of the type (4.15), generally speaking, cannot be solved in a simple manner. Depending on the ratios between the elastic constants of the plate material, different variations in the roots of the characteristic equation are possible, and, therefore, different types of solutions of the initial system of equations are obtained. Nevertheless, procedures introduced in this section remain unchanged and are valid for any such variation of the solution of the initial system of equations.

The case of the transversely isotropic plate with transverse bending may be more simply solved with equations (2.25) and (2.26) as will be shown in the next section.

5. Bending of a Semi-Infinite Plate by a Load Distributed Along the Edge

Consider the transverse bending of a transversely isotropic semi-infinite plate, the middle plane of which occupies the entire half-plane $x \geqslant 0$, bent by a bending moment M_x, a twisting moment H, and transverse shear force N_x applied along the edge $x = 0$ [24] (Figure 15).

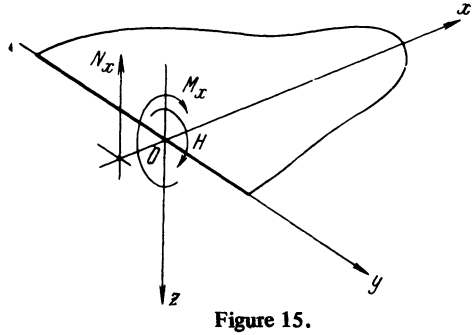

Figure 15.

Let the edge loads be distributed in the following manner:

$$\left.\begin{aligned} M_x &= M_0 \sin\frac{\pi y}{\lambda}, \\ H &= H_0 \cos\frac{\pi y}{\lambda}, \\ N_x &= N_0 \sin\frac{\pi y}{\lambda}, \end{aligned}\right\} \quad (5.1)$$

where M_0, H_0 and N_0 are constants; λ is the length between changes in the direction (sign) of the applied loads along the plate edge at $x = 0$. It is assumed that λ is sufficiently large in comparison with the plate thickness h.

Let us start with equations (2.25) and (2.26) which can be expressed in the following manner because of the absence of external surface load ($Z = 0$):

$$\left.\begin{aligned} \Delta\Delta w &= 0, \\ \Delta\Phi - \delta^2\Phi &= 0; \end{aligned}\right\} \tag{5.2}$$

where

$$\delta^2 = \frac{10}{h^2}\frac{G'}{G}. \tag{5.3}$$

Substituting values of φ and ψ from (2.27) and (2.28) into (2.15), respectively, we obtain for moment resultants and transverse shear resultants:

$$M_x = -D\Delta w + D(1-\nu)\frac{\partial^2 w}{\partial y^2} + \frac{2}{\delta^2}D\frac{\partial^2}{\partial y^2}\Delta w + \frac{2}{\delta^2}\frac{\partial^2\Phi}{\partial x\,\partial y}, \tag{5.4}$$

$$M_y = -D\Delta w + D(1-\nu)\frac{\partial^2 w}{\partial x^2} + \frac{2}{\delta^2}D\frac{\partial^2}{\partial x^2}\Delta w - \frac{2}{\delta^2}\frac{\partial^2\Phi}{\partial x\,\partial y}, \tag{5.5}$$

$$H = -D(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y} - \frac{2}{\delta^2}D\frac{\partial^2}{\partial x\,\partial y}\Delta w + \Phi - \frac{2}{\delta^2}\frac{\partial^2\Phi}{\partial x^2}, \tag{5.6}$$

$$N_x = -D\frac{\partial}{\partial x}\Delta w + \frac{\partial\Phi}{\partial y}, \tag{5.7}$$

$$N_y = -D\frac{\partial}{\partial y}\Delta w - \frac{\partial\Phi}{\partial x}. \tag{5.8}$$

During the derivation of (5.4) – (5.8) the first equation from (5.2) was frequently used.

The solution of the system of equations (5.2) is sought in the form:

$$w = W(x)\sin\alpha y, \qquad \Phi = \Phi_0(x)\cos\alpha y, \tag{5.9}$$

where

$$\alpha = \frac{\pi}{\lambda}. \tag{5.10}$$

Substituting expressions (5.9) into (5.2), the following ordinary differential equations with respect to W and Φ are obtained:

$$\left.\begin{aligned} W^{IV} - 2\alpha^2 W'' + \alpha^4 W &= 0, \\ \Phi_0'' - (\alpha^2 + \delta^2)\Phi_0 &= 0. \end{aligned}\right\} \tag{5.11}$$

The general solution of the system (5.11) is

$$\left.\begin{aligned} W(x) &= (c_1 + c_2 x)e^{-\alpha x} + (c_3 + c_4 x)e^{\alpha x}, \\ \Phi_0(x) &= c_5 e^{-\alpha\omega x} + c_6 e^{\alpha\omega x}, \end{aligned}\right\} \tag{5.12}$$

where

$$\omega = \sqrt{1 + \frac{\delta^2}{\alpha^2}}. \tag{5.13}$$

Here the c_i are integration constants which must be determined from the boundary conditions of the problem. Considering that the moments and stresses should approach zero as x becomes large, we assume that

$$c_3 = c_4 = c_6 = 0. \tag{5.14}$$

Finally, due to (5.14), (5.12) and (5.9) the solution may be written:

$$w = (c_1 + c_2 x)e^{-\alpha x} \sin \alpha y, \tag{5.15}$$

$$\Phi = c_5 e^{-\alpha\omega x} \cos \alpha y. \tag{5.16}$$

Thus, the problem under consideration is reduced to the determination of three integration constants, c_1, c_2, c_5. These constants are found from the conditions at the edge x = 0, i.e., from the conditions (5.1). Generally speaking, some of the load conditions (5.1) can be replaced by geometric conditions for the consideration of other boundary value problems.

Particular cases are discussed below.

1. Edge of the semi-infinite plate loaded by forces which change according to some particular law. On the basis of expressions (5.15), (5.16), and also (5.4) – (5.7), the conditions at the edge x = 0 are

$$\left.\begin{aligned} (1-\nu)\alpha c_1 - 2\left(1 + 2\frac{\alpha^2}{\delta^2}\right)c_2 - 2\omega\frac{\alpha^2}{\delta^2}\frac{1}{D\alpha}c_5 &= -\frac{M_0}{D\alpha}, \\ (1-\nu)\alpha c_1 - \left[1 - \nu + 4\frac{\alpha^2}{\delta^2}\right]c_2 - \left(1 + 2\frac{\alpha^2}{\delta^2}\right)\frac{1}{D\alpha}c_5 &= \frac{H_0}{D\alpha}, \\ 2c_2 + \frac{1}{D\alpha}c_5 &= -\frac{N_0}{D\alpha}. \end{aligned}\right\} \tag{5.17}$$

It is evident that the determinant of the system (5.17) is not zero and, therefore, definite values of the constants c_1, c_2 and c_5 can be obtained; thus the total solution can be explicitly obtained.

Consider two particular cases.

Case I. Only bending moment M_x acts at the edge x = 0. It is assumed in this case that $H_0 = N_0 = 0$. Then from the system (5.17)

$$c_1 = -\frac{M_0(1+\nu)}{D(1-\nu)\alpha^2}\frac{1}{3+\nu-4\frac{\alpha^2}{\delta^2}(\omega-1)},$$

$$c_2 = \frac{M_0}{D\alpha}\frac{1}{3+\nu-4\frac{\alpha^2}{\delta^2}(\omega-1)},$$

$$c_5 = -\frac{2M_0}{3+\nu-4\frac{\alpha^2}{\delta^2}(\omega-1)}.$$

The normal displacement w(x, y), is obtained from equality (5.15):

$$w = w_0\left[1-\frac{4}{3+\nu}\frac{t}{t+\sqrt{1+t^2}}\right]^{-1}, \tag{5.18}$$

where w_0 is the plate deflection as found from the classical theory, namely

$$w_0 = -\frac{M_0\lambda^2}{\pi^2(3+\nu)D}\left(\frac{1+\nu}{1-\nu}-\frac{\pi}{\lambda}x\right)e^{\frac{-\pi x}{\lambda}}\sin\frac{\pi y}{\lambda},$$

and t is the relative thickness

$$t = \frac{\pi h}{2\lambda}\sqrt{\frac{E}{5(1+\nu)G'}}. \tag{5.19}$$

It is evident that the result of the classical theory is obtained at t = 0.

Expansion of the right-hand side of (5.18) into the Maclaurin series and keeping only the first three terms we obtain

$$w = w_0\left[1+\frac{4}{(3+\nu)}t+\frac{4(1-\nu)}{(3+\nu)^2}t^2\right]. \tag{5.20}$$

The extent of the correction to the classical theory is easily seen from (5.20) to contain t in the first power.

Case II. Only the transverse shear force N_x is acting at the edge x = 0. It is assumed here that $M_0 = H_0 = 0$. Then from the system (5.17) we have

$$c_1 = -\frac{2N_0}{D(1-\nu)\alpha^3}\frac{(1+2t^2)^2-\omega t^2(1-\nu+4t^2)}{3+\nu-4t^2(\omega-1)},$$

$$c_2 = -\frac{N_0}{D\alpha^2}\frac{1-2t^2(\omega-1)}{3+\nu-4t^2(\omega-1)},$$

$$c_5 = -\frac{N_0}{\alpha}\frac{1+\nu}{3+\nu-4t^2(\omega-1)}.$$

On the basis of equation (5.15) the deflection w is

$$w = -\frac{N_0\lambda^2}{\pi^2 D}\left\{\frac{2\lambda}{\pi(1-\nu)}[(2t^2+1)^2 - t(4t^2+1-\nu)\sqrt{t^2+1}] + \right.$$
$$\left. + (\sqrt{t^2+1}-t)^2 x\right\} \frac{e^{-\pi x/\lambda}\sin\dfrac{\pi y}{\lambda}}{3+\nu-4t(\sqrt{t^2+1}-t)}. \quad (5.21)$$

When t = 0 the corresponding expression for w_0 of the classical theory is derived:

$$w_0 = -\frac{N_0\lambda^2}{\pi^2(3+\nu)D}\left[\frac{2\lambda}{\pi(1-\nu)} + x\right] e^{-\frac{\pi x}{\lambda}} \sin\frac{\pi y}{\lambda}. \quad (5.22)$$

2. Edge of the semi-infinite plate simply-supported and loaded by a bending moment M_x. The conditions of simple support are defined in Section 3 of Chapter II by formulas (2.3.14) and (2.3.14'). We will attempt to solve this problem using two variations of the simple support boundary conditions to see how the theory suggested in this book behaves, in this case, for these two different assumptions.

Variation I. The boundary conditions (2.3.14) for this problem are:

at x = 0

$$w = 0, \qquad M_x = M_0 \sin \alpha y, \qquad H = 0. \quad (5.23)$$

On the basis of (5.15) and (5.16), as well as equations (5.4) and (5.6) the conditions (5.23) are

$$\left.\begin{aligned} c_1 &= 0, \\ (1-\nu)\alpha c_1 - 2\left(1+2\frac{\alpha^2}{\delta^2}\right)c_2 - 2\omega\frac{\alpha^2}{\delta^2}\frac{1}{D\alpha}c_5 &= -\frac{M_0}{D\alpha}, \\ (1-\nu)\alpha c_1 - \left[1-\nu+4\frac{\alpha^2}{\delta^2}\right]c_2 - \left(1+2\frac{\alpha^2}{\delta^2}\right)\frac{1}{D\alpha}c_5 &= 0. \end{aligned}\right\} \quad (5.24)$$

Solving the system (5.24) for the integration constants c_i we obtain

$$c_1 = 0, \qquad c_2 = \frac{M_0}{2D\alpha}\frac{1+2\dfrac{\alpha^2}{\delta^2}}{\left(1+2\dfrac{\alpha^2}{\delta^2}\right)^2 - \omega\dfrac{\alpha^2}{\delta^2}\left(1-\nu+4\dfrac{\alpha^2}{\delta^2}\right)},$$

$$c_5 = -\frac{M_0}{2}\frac{1-\nu+4\dfrac{\alpha^2}{\delta^2}}{\left(1+2\dfrac{\alpha^2}{\delta^2}\right)^2 - \omega\dfrac{\alpha^2}{\delta^2}\left(1-\nu+4\dfrac{\alpha^2}{\delta^2}\right)},$$

and for the functions w and Φ we have

$$w = \frac{M_0}{2D\alpha} \frac{\left(1 + 2\frac{\alpha^2}{\delta^2}\right) x e^{-\alpha x} \sin \alpha y}{\left(1 + 2\frac{\alpha^2}{\delta^2}\right)^2 - \omega \frac{\alpha^2}{\delta^2}\left(1 - \nu + 4\frac{\alpha^2}{\delta^2}\right)}, \tag{5.25}$$

$$\Phi = -\frac{M_0}{2} \frac{\left(1 - \nu + 4\frac{\alpha^2}{\delta^2}\right) e^{-\alpha\omega x} \cos \alpha y}{\left(1 + 2\frac{\alpha^2}{\delta^2}\right)^2 - \omega \frac{\alpha^2}{\delta^2}\left(1 - \nu + 4\frac{\alpha^2}{\delta^2}\right)}. \tag{5.26}$$

After the determination of these functions and using equations (5.4) – (5.8), the moments and transverse shears for the problem can be determined.

Variation II. Boundary conditions (2.3.14′) are in this case at x = 0

$$w = 0, \qquad M_x = M_0 \sin \alpha y, \qquad \psi = 0. \tag{5.27}$$

On the basis of (5.15), (5.16), (5.4) and (2.27), the conditions (5.27) are

$$\left.\begin{aligned} c_1 = 0, \qquad 2\alpha c_2 + \frac{\omega}{D} c_5 = 0, \\ (1 - \nu)\alpha c_1 - 2\left(1 + 2\frac{\alpha^2}{\delta^2}\right) c_2 - 2\omega \frac{\alpha^2}{\delta^2} \frac{1}{D\alpha} c_5 = -\frac{M_0}{D\alpha}. \end{aligned}\right\} \tag{5.28}$$

Solving the system (5.28) for the integration constants c_i we have

$$c_1 = 0, \qquad c_2 = \frac{M_0}{2D\alpha}, \qquad c_5 = -\frac{M_0}{\omega},$$

and for functions w and Φ:

$$w = \frac{M_0}{2D\alpha} x e^{-\alpha x} \sin \alpha y, \tag{5.29}$$

$$\Phi = -\frac{M_0}{\omega} e^{-\alpha\omega x} \cos \alpha y. \tag{5.30}$$

This second variation is completely determined using (5.29), (5.30) and (5.4) – (5.8).

Comparing expressions (5.29) and (5.30) with (5.25) and (5.26) we see that these variations of the simple support differ significantly from each other and this difference influences the final results. In reality we actually have here two different boundary conditions. Nevertheless, considering that the classical conditions of simple supports (conditions for w and M) are present for both these variations of simple supports, it is consistent at

this point to combine them under one heading. These boundary conditions may, however, be represented under different headings as more information is accumulated in the process of this study. For example, conditions (2.3.14) might continue to be considered as the conditions of a simply-supported edge, and conditions (2.3.14′) with the limitation on ψ might be considered as the conditions of the simply-fixed edge. Thus, we can say that the theory treated in this book is sensitive to both variations of simple support. During the analysis of plates with simply-supported edges it is important to carefully select the corresponding mathematical model, considering, of course, the actual type of plate conditions which are to be simulated.

6. Axisymmetric Bending of a Circular Curvilinear Anisotropic Plate

Consider a circular plate of radius a, made of material having cylindrical anisotropy (Figure 16). The origin of coordinates (r, θ, z) coincides with the circular plate center which also represents the pole of anisotropy. The plate is loaded by a normally applied uniformly distributed load of intensity q in such a manner that

$$Z^+ = q, \qquad Z^- = 0 \tag{6.1}$$

and on the basis of (2.2.29) we have

$$Z_1 = \frac{q}{2}, \qquad Z_2 = q. \tag{6.2}$$

Accepting further that we have axisymmetrical boundary conditions (with respect to the axis of anisotropy, i.e., the z-axis) it is not difficult to see that the deformed middle surface of the plate also represents a surface of revolution with the rotation taken about the z-axis.

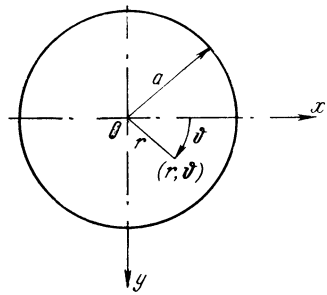

Figure 16.

Then from (2.7.23) – (2.7.27) the following system of differential equations is obtained:

$$C_r\left(r\frac{d^2u}{dr^2}+\frac{du}{dr}\right)-C_\vartheta\frac{u}{r}=\frac{qh}{2}(A_1-A_2), \tag{6.3}$$

$$\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}-\frac{v}{r^2}=0, \tag{6.4}$$

$$\frac{1}{r}\frac{d}{dr}(r\varphi)=-\frac{12}{h^3}q, \tag{6.5}$$

$$D_r\frac{d}{\mathrm{dr}}\left(r\frac{d^2w}{dr^2}\right)-D_\vartheta\frac{1}{r}\frac{dw}{dr}+\frac{h^2}{10}a_r\left[D_\vartheta\frac{\varphi}{r}-D_r\frac{d}{dr}\left(r\frac{d\varphi}{dr}\right)\right]+$$
$$+\frac{h^3}{12}r\varphi=\frac{h^2}{10}(A_2-A_1)q, \tag{6.6}$$

$$\frac{d^2\psi}{dr^2}+\frac{1}{r}\frac{d\psi}{dr}-\left(\frac{1}{r^2}+\frac{5h}{6a_\vartheta D_k}\right)\psi=0. \tag{6.7}$$

Equations (6.3) – (6.7) together with the boundary conditions constitute a complete system with respect to five undetermined functions, u(r), v(r), w(r), φ(r) and ψ(r).

On the basis of (2.7.6) – (2.7.16) the internal stress and moment resultants and the stresses are:

$$\sigma_r=B_r\left(\frac{du}{dr}-z\frac{d^2w}{dr^2}\right)+B_r\nu_\vartheta\left(\frac{u}{r}-z\frac{1}{r}\frac{dw}{dr}\right)+$$
$$+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)B_ra_r\left(\frac{d\varphi}{dr}+\nu_\vartheta\frac{\varphi}{r}\right)-$$
$$-A_1\left[\left(\frac{3}{2}\frac{z}{h}-2\frac{z^3}{h^3}\right)+\frac{1}{2}\right]q, \tag{6.8}$$

$$\sigma_\vartheta=B_\vartheta\left(\frac{u}{r}-z\frac{1}{r}\frac{dw}{dr}\right)+B_\vartheta\nu_r\left(\frac{du}{dr}-z\frac{d^2w}{dr^2}\right)+$$
$$+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)B_\vartheta a_r\left(\frac{\varphi}{r}+\nu_r\frac{d\varphi}{dr}\right)-$$
$$-A_2\left[\left(\frac{3}{2}\frac{z}{h}-2\frac{z^3}{h^3}\right)+\frac{1}{2}\right]q, \tag{6.9}$$

$$\tau_{\vartheta r}=B_k\left[\frac{dv}{dr}-\frac{v}{r}+\frac{z}{2}\left(\frac{h^2}{4}-\frac{z^2}{3}\right)a_\vartheta\left(\frac{d\psi}{dr}-\frac{\psi}{r}\right)\right], \tag{6.10}$$

$$T_r=C_r\frac{du}{dr}+C_r\nu_\vartheta\frac{u}{r}-hA_1\frac{q}{2}, \tag{6.11}$$

$$T_\vartheta=C_\vartheta\frac{u}{r}+C_\vartheta\nu_r\frac{du}{dr}-hA_2\frac{q}{2}, \tag{6.12}$$

$$S_{\vartheta r}=C_k\left(\frac{dv}{dr}-\frac{v}{r}\right), \tag{6.13}$$

$$M_r = -D_r \frac{d^2w}{dr^2} - D_r \nu_\vartheta \frac{1}{r}\frac{dw}{dr} + \frac{h^2}{10} D_r a_r \left(\frac{d\varphi}{dr} + \nu_\vartheta \frac{\varphi}{r}\right) - \frac{h^2}{10} A_1 q, \quad (6.14)$$

$$M_\vartheta = -D_\vartheta \frac{1}{r}\frac{dw}{dr} - D_\vartheta \nu_r \frac{d^2w}{dr^2} + \frac{h^2}{10} D_\vartheta a_r \left(\frac{\varphi}{r} + \nu_r \frac{d\varphi}{dr}\right) - \frac{h^2}{10} A_2 q, \quad (6.15)$$

$$H_{r\vartheta} = D_k a_\vartheta \frac{h^2}{10}\left(\frac{d\psi}{dr} - \frac{\psi}{r}\right), \quad (6.16)$$

$$N_r = \frac{h^3}{12}\varphi, \qquad N_\vartheta = \frac{h^3}{12}\psi. \quad (6.17)$$

Equations (6.3) and (6.4) describe the plane problem.

Considering that v = 0 in the center of the axisymmetrically deformed plate and eliminating a rigid body rotation of the plate with respect to the z-axis (which is present in the solution of (6.4)), we have for the tangential displacement v;

$$v = 0. \quad (6.18)$$

Due to (6.18) and using (6.13) we also have that:

$$S_{\vartheta r} = 0. \quad (6.19)$$

The second equation of the plane problem (6.3) can be rearranged to this form:

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - n^2\frac{u}{r^2} = \frac{qh}{2r}\frac{A_1 - A_2}{C_r},$$

where

$$n^2 = \frac{E_\vartheta}{E_r} = \frac{C_\vartheta}{C_r} = \frac{D_\vartheta}{D_r}. \quad (6.20)$$

A simple integration of equation (6.3) for $n \neq 1$ gives:

$$u = c_1 r^n + c_2 r^{-n} + \frac{A_1 - A_2}{C_r(1 - n^2)}\frac{qh}{2} r.$$

The integration constant c_2, on the basis of the conditions at the plate center (u = 0 at r = 0) is equal to zero. Then for the radial displacement we have

$$u = c_1 r^n + \frac{qh(A_1 - A_2)}{2C_r(1 - n^2)} r, \quad (6.21)$$

The integration constant c_1 can be determined from conditions at the plate edge.

Equation (6.5) is easily solved for the function φ:

$$\varphi = -\frac{12}{h^3}\frac{q}{2} r + \frac{c_3}{r}.$$

The integration constant c_3 is equal to zero as can be determined from the conditions on φ at the plate center. Then for φ we have:

$$\varphi = -\frac{12}{h^3}\frac{q}{2}r. \tag{6.22}$$

Substituting this value of φ from (6.22) into equation (6.6), the following differential equation is obtained for w:

$$\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{n^2}{r^2}\frac{dw}{dr} = \frac{qr}{2D_r} + \left[(A_2 - A_1)\frac{h^2}{10D_r} + \frac{3a_r}{5h}(n^2 - 1)\right]\frac{q}{r}. \tag{6.23}$$

The solution of equation (6.23) for $n \neq 1$ is

$$w = c_4 + c_5 r^{1+n} + c_6 r^{1-n} + b_1 r^2 + b_2 r^4, \tag{6.24}$$

where

$$b_1 = \frac{qh^2}{20(1-n^2)}\left[\frac{6a_r}{h^3}(n^2 - 1) + \frac{A_2 - A_1}{D_r}\right], \tag{6.25}$$

$$b_2 = \frac{q}{8(9-n^2)D_r}. \tag{6.26}$$

Due to the fact that the center of the deformed plate has zero slope (at r = 0, dw/dr = 0) the constant c_6 is assumed to be equal to zero.

Thus, the solution for w is:

$$w = c_4 + c_5 r^{1+n} + b_1 r^2 + b_2 r^4. \tag{6.27}$$

The integration constants c_4 and c_5 are determined from the conditions at the plate edge.

Equation (6.7) can be represented as follows

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} - \left(\frac{1}{r^2} + \alpha^2\right)\psi = 0, \tag{6.28}$$

where

$$\alpha^2 = \frac{5h}{6a_\theta\ D_k}$$

It is easy to see that equation (6.28) represents the Bessel equation and its solution can thus be derived with the Bessel function of the first and second kind of imaginary argument, namely [25]

$$\psi = c_7 I_1(\alpha r) + c_8 K_1(\alpha r).$$

Due to the limitation on ψ at the plate center the integration constant c_8 is equal to zero; then for ψ we have

$$\psi = c_7 I_1(\alpha r), \tag{6.29}$$

The integration constant c_7 is determined from the conditions at the plate edge, just as in all the other cases.

We will have need in this section for the functions and some of their derivatives. Below these expressions are given for convenience:

$$\left.\begin{aligned} u = c_1 r^n + \frac{qh(A_1 - A_2)}{2C_r(1-n^2)} r, \quad \frac{du}{dr} = c_1 n r^{n-1} + \frac{qh(A_1 - A_2)}{2C_r(1-n^2)}, \\ w = c_4 + c_5 r^{1+n} + b_1 r^2 + b_2 r^4, \\ \frac{dw}{dr} = c_5(1+n) r^n + 2b_1 r + 4b_2 r^3, \end{aligned}\right\} \quad (6.30)$$

$$\left.\begin{aligned} \frac{d^2w}{dr^2} = c_5(1+n) n r^{n-1} + 2b_1 + 12 b_2 r^2, \\ \varphi = -\frac{12}{h^3}\frac{q}{2} r, \quad \frac{d\varphi}{dr} = -\frac{12}{h^3}\frac{q}{2}, \\ \psi = c_7 I_1(\alpha r), \quad \frac{d\psi}{dr} = c_7\left(\frac{1}{r} I_1(\alpha r) + \alpha I_2(\alpha r)\right). \end{aligned}\right\} \quad (6.30)$$

We now consider particular boundary conditions along the plate contour.

1. Plate with a simply-supported contour. In the case of a simply-supported plate we have from equations (2.3.14) or (2.3.20) at r = a

$$T_r = 0, \qquad M_r = 0, \qquad H_{\vartheta r} = 0, \qquad w = 0. \qquad (6.31)$$

From equations (6.11), (6.14), (6.16) and considering (6.21), (6.22), (6.24), (6.29) and (6.30), the boundary conditions (6.31) are easily satisfied, and as a result we obtain expressions for the integration constants:

$$c_1 = \frac{qha^{1-n}}{2C_r(n+\nu_\vartheta)}\left[A_1 - \frac{(A_1 - A_2)(1+\nu_\vartheta)}{2(1-n^2)}\right],$$

$$c_4 = \frac{qh^2a^2}{10(1-n^2)}\left[\frac{A_1(n-n^2-\nu_\vartheta+n\nu_\vartheta) + A_2(2-n-n^2+\nu_\vartheta-n\nu_\vartheta)}{2D_r(1+n)(n+\nu_\vartheta)} + \right.$$
$$\left. + \frac{3a_r(1-n^2)}{h^3}\right] + \frac{qa^4}{8D_r(9-n^2)}\left[\frac{4(3-\nu_\vartheta)}{(1+n)(n+\nu_\vartheta)} - 1\right],$$

$$c_5 = -\frac{q}{D_r(1+n)(n+\nu_\vartheta)}\left\{\frac{h^2 a^{1-n}}{10(1-n^2)}[A_2(1+\nu_\vartheta) - \right.$$
$$\left. - A_1(n^2+\nu_\vartheta)] + \frac{(3+\nu_\vartheta)a^{3-n}}{2(9-n^2)}\right\},$$
$$c_7 = 0.$$

Substituting these values of the integration constants c_1, c_4, c_5 and c_7 into the corresponding equations, final expressions for the functions u, w, and ψ are obtained. These expressions are not given here because of the simplicity of the procedures. However, it should be indicated that $\psi = 0$ for the boundary conditions considered here. Expressions for other pertinent quantities can easily be derived from (6.8) - (6.17) by substituting the solutions for u, v, w, φ and ψ.

In the particular case of the transversely isotropic plate when the plane of isotropy of the material at each point is parallel to the plate middle plane we obtain the following expressions on the basis of (2.10) or, more completely, because of (1.4.10), (1.4.11), (2.5.1), (2.5.2), (2.7.9), (2.7.17) and (6.20):

$$\left.\begin{aligned}
&B_r = B_\vartheta = B_{r\vartheta} = B_r\nu + 2B_k = \frac{E}{1-\nu^2},\\
&B_k = G_{r\vartheta} = \frac{E}{2(1+\nu)}, \qquad a_r = a_\vartheta = \frac{1}{G'},\\
&\nu_\vartheta = \nu_r = \nu, \qquad -\frac{\nu_{rz}}{E_z} = -\frac{\nu_{\vartheta z}}{E_z} = -\frac{\nu'}{E'},\\
&A_1 = A_2 = A = -\frac{\nu'}{(1-\nu)}\frac{E}{E'}, \qquad n = 1,\\
&D = \frac{Eh^3}{12(1-\nu^2)}.
\end{aligned}\right\} \tag{6.32}$$

On the basis of (6.32) and using (6.8) – (6.17) the stresses, the stress resultants, and the moment resultants are:

$$\sigma_r = \frac{E}{1-\nu^2}\left[\frac{du}{dr} - z\frac{d^2w}{dr^2} + \frac{z}{2G'}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\frac{d\varphi}{dr}\right] +$$
$$+ \frac{\nu E}{1-\nu^2}\left[\frac{u}{r} - \frac{z}{r}\frac{dw}{dr} + \frac{z}{2G'}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\frac{\varphi}{r}\right] - A\left(\frac{1}{2} + \frac{3z}{2h} - 2\frac{z^3}{h^3}\right)q, \tag{6.33}$$

$$\sigma_\vartheta = \frac{E}{1-\nu^2}\left[\frac{u}{r} - \frac{z}{r}\frac{dw}{dr} + \frac{z}{2G'}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\frac{\varphi}{r}\right] +$$
$$+ \frac{\nu E}{1-\nu^2}\left[\frac{du}{dr} - z\frac{d^2w}{dr^2} + \frac{z}{2G'}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\frac{d\varphi}{dr}\right] - A\left(\frac{1}{2} + \frac{3z}{2h} - 2\frac{z^2}{h^3}\right)q, \tag{6.34}$$

$$T_r = \frac{Eh}{1-\nu^2}\frac{du}{dr} + \frac{\nu Eh}{1-\nu^2}\frac{u}{r} - hA\frac{q}{2}, \tag{6.35}$$

$$T_\vartheta = \frac{Eh}{1-\nu^2}\frac{u}{r} + \frac{\nu Eh}{1-\nu^2}\frac{du}{dr} - hA\frac{q}{2}, \tag{6.36}$$

$$M_r = -\frac{Eh^3}{12(1-\nu^2)}\left(\frac{d^2w}{dr^2} + \nu\frac{dw}{r\,dr}\right) - A\frac{h^2}{10}q +$$
$$+ \frac{h^2}{10}\frac{Eh^3}{12G'(1-\nu^2)}\left(\frac{d\varphi}{dr} + \nu\frac{\varphi}{r}\right), \tag{6.37}$$

$$M_\vartheta = -\frac{Eh^3}{12(1-\nu^2)}\left(\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2}\right) - A\frac{h^2}{10}q +$$
$$+ \frac{h^2}{10}\frac{Eh^3}{12(1-\nu^2)G'}\left(\frac{\varphi}{r} + \nu\frac{d\varphi}{dr}\right), \tag{6.38}$$

$$N_r = \frac{h^3}{12}\varphi, \tag{6.39}$$

The functions u, w, and φ present in (6.33) – (6.39) are given by (6.22) and the following final results [26]:

$$u = -\frac{\nu'}{E'}\frac{qr}{2}, \qquad \psi = 0, \tag{6.40}$$

$$w = \frac{qa^4}{64D}\left[\frac{r^4}{a^4} - 2\frac{3+\nu}{1+\nu}\frac{r^2}{a^2} + \frac{5+\nu}{1+\nu}\right] +$$
$$+ \frac{qh^2}{20D(1-\nu)}\left[\frac{G}{G'} - \frac{\nu'}{(1+\nu)}\frac{E}{E'}\right](a^2 - r^2). \tag{6.41}$$

Substituting these values of φ, u and w from (6.22), (6.40) and (6.41), respectively, into (6.33) – (6.39), formulas for the stress and moment resultants and stresses can be easily obtained.

Equation (6.41) can also be easily derived from the solutions of (6.27) and (6.22) by passage to the limit while considering the values of the

corresponding constants b_i and c_i. However, it is simpler to derive equation (6.41) directly from (6.23) and the boundary conditions (6.31) for the transversely isotropic plate.

From (6.41) the deflection of the center of the plate (r = 0) is:

$$w_c = \frac{qa^4}{64D}\frac{5+\nu}{1+\nu}\left\{1+\frac{16h^2(1+\nu)}{5(1-\nu)(5+\nu)a^2}\left[\frac{G}{G'}-\frac{\nu'}{(1+\nu)}\frac{E}{E'}\right]\right\} \tag{6.42}$$

where the first term $\dfrac{qa^4}{64\ D}\dfrac{(5+\nu)}{(1+\nu)}$

is the deflection of the plate center as computed from the classical theory [27].

If this problem is solved with the simplified theory which disregards the effect of normal stress σ_z, then from expressions (6.22), (6.40) and (6.41) we can obtain, evidently,

$$\varphi = -\frac{12}{h^3}\frac{q}{2}r, \qquad u = 0, \tag{6.43}$$

$$w_c = \frac{qa^4}{64D}\left[\frac{r^4}{a^4}-2\frac{3+\nu}{1+\nu}\frac{r^2}{a^2}+\frac{5+\nu}{1+\nu}\right]+ \\ +\frac{qh^2}{20D(1-\nu)}\frac{G}{G'}(a^2-r^2), \tag{6.44}$$

Equations (6.33) – (6.39) are available to determine the other quantities of interest using equations (6.43) and (6.44) and assuming that A = 0. In this case the deflection of the center of the plate is obtained from (6.42) and (6.44):

$$w_c = \frac{qa^4}{64D}\frac{5+\nu}{1+\nu}\left[1+\frac{16(1+\nu)h^2G}{5(1-\nu)(5+\nu)a^2G'}\right] \tag{6.45}$$

This same problem of the bending of a circular plate under a uniformly distributed load with a simply-supported contour can be solved with the use of the particular theory. Without mentioning any details it should be indicated that all the pertinent results obtained in this way are completely identical with the corresponding formulas derived with the use of the generalized theory, i.e., the simplified general theory [28] which disregards the effect of σ_z.

Using the same circular plate it can be shown that the theories suggested in this book (general and particular) can be applied to the analysis of thick plates for certain boundary conditions and loads. As an example, consider the axi-symmetrical bending of a circular isotropic plate simply-supported along the contour and carrying a uniformly distributed load of intensity q.

An accurate solution for the deflection of the plate middle plane leads to the following expression [29]

$$w_c = \frac{qa^4}{64D}\frac{5+\nu}{1+\nu}\left[1+\frac{2(8+\nu+\nu^2)h^2}{5(1-\nu)(5+\nu)a^2}\right]. \tag{6.46}$$

According to the theory suggested here and using equations (6.42) and (6.45) for the isotropic plate (G = G', E = E', $\nu = \nu'$) the center deflection is

$$w_c = \frac{qa^4}{64D}\frac{5+\nu}{1+\nu}\left[1+\frac{16}{5(1-\nu)(5+\nu)}\frac{h^2}{a^2}\right], \tag{6.47}$$

and if the effect of σ_z is disregarded

$$w_c = \frac{qa^4}{64D}\frac{5+\nu}{1+\nu}\left[1+\frac{16(1+\nu)}{5(1-\nu)(5+\nu)}\frac{h^2}{a^2}\right]. \tag{6.48}$$

Comparing equations (6.46) with (6.47) and (6.48) we see that all of the equations are identical when Poisson's ratio is equal to zero ($\nu = 0$). Therefore, we need a numerical comparison only in the case when Poisson's ratio is larger than zero. For example, if we assume that h/a = 1/3 and $\nu = 0.3$, we obtain the results shown in the table below.

	From Accurate Formula	From (6.47)	From (6.48)	From the Classical Theory of Plates
$\frac{64D\ (1+\nu)}{qa^4\ (5+\nu)}w$	1.1005	1.0958	1.1245	1.0000

2. Plate with a clamped contour. In the case of a plate with a clamped contour from (2.3.15) or (2.3.21) we have at r = a because of (2.4.1):

$$\left.\begin{aligned} u = 0, \quad w = 0, \quad \psi = 0, \\ -\frac{dw}{dr} + a_r\frac{1}{2}\left(\frac{h^2}{4}-\frac{z_0^2}{3}\right)\varphi = 0. \end{aligned}\right\} \tag{6.49}$$

On the basis of (6.21), (6.22), (6.24), (6.30) and using the boundary conditions (6.49) we obtain the integration constants:

$$c_1 = -\frac{qh(A_1-A_2)}{2C_r(1-n^2)}a^{1-n}, \quad c_7 = 0,$$

$$c_4 = \frac{qa^2}{(1+n)}\left[\frac{(3-n)a^2}{8(9-n^2)D_r}+\frac{3a_r}{h^3}\left(\frac{h^2}{4}-\frac{z_0^2}{3}\right)-\right.$$

$$\left.-\frac{3a_r(1-n)}{10h}+\frac{(A_2-A_1)h^2}{20(1+n^2)D_r}\right],$$

$$c_5 = -\frac{2qa^{1-n}}{(1+n)}\left[\frac{a^2}{4(9-n^2)D_r}+\frac{3a_r}{2h^3}\left(\frac{h^2}{4}-\frac{z_0^2}{3}\right)-\right.$$
$$\left.-\frac{3a_r}{10h}+\frac{(A_2-A_1)h^2}{20(1-n^2)D_r}\right]$$

Substituting the values of the integration constants c_1, c_4, c_5 and c_7 into the corresponding expressions (6.21), (6.27), (6.29) and (6.30), final expressions are obtained for the functions u, w and ψ. The expressions for these functions as well as for the other pertinent quantities (which can easily be obtained from (6.8) – (6.17)) are not given here.

In the particular case of a transversely isotropic plate when the plane of isotropy of the material is parallel to the plate middle plane, equations (6.32) – (6.39) are available for the material constants, the stress and moment resultants, and the stresses. For the functions present in (6.33) – (6.39) we have

$$u = 0, \quad \psi = 0, \quad \varphi = -\frac{12}{h^3}\frac{q}{2}r, \tag{6.50}$$

$$w = \frac{q}{64D}(a^2-r^2)^2+\frac{3q}{8hG'}(a^2-r^2). \tag{6.51}$$

During the derivation of (6.51) it was assumed that $z_0 = 0$, i.e., all vertical elements of the plate edge adjacent to the middle plane were considered to be fixed. A detailed analysis of the effect of z_0 on the plate deflection can be found in Section 1 of this chapter. It is of interest to notice that the value of w given in (6.51) is identical to the precise solution [30].

The plate with clamped contour can also be solved with the use of particular theory. The results obtained in this case, as expected, would not differ from those obtained from the general theory [31].

7. Two Problems of Bending of Transversely Isotropic Plates in a Cylindrical Coordinate System

The system of differential equations (2.7.25) – (2.7.27) for the case of a transversely isotropic plate with the plane of isotropy parallel to the middle plane at each point of the plate can be reduced to two independent differential equations with respect to two functions $w(r, \theta)$ and $\Phi(r, \theta)$ after certain simple transformations:

$$D\Delta\,\Delta w = Z - k\,\Delta Z, \tag{7.1}$$

$$\Delta\Phi - \delta^2\Phi = 0; \tag{7.2}$$

The following notations are used here

$$\Delta = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial}{\partial \vartheta}\left(\frac{1}{r}\frac{\partial}{\partial \vartheta}\right)\right], \tag{7.3}$$

$$k = \left(2\frac{G}{G'} - \nu'\frac{E}{E'}\right)\frac{h^2}{10(1-\nu)}, \quad \delta^2 = \frac{10}{h^2}\frac{G'}{G}, \tag{7.4}$$

$$Z_2 = Z = Z^+ + Z^-, \quad Z_1 = \frac{1}{2}(Z^+ - Z^-) = 0. \tag{7.5}$$

The undetermined function $\Phi(r, \theta)$ is related to the functions $w(r, \theta)$, $\varphi(r, \theta)$ and $\psi(r, \theta)$ by these expressions:

$$\frac{h^3}{12}\varphi = -D\frac{\partial}{\partial r}(\Delta w)+\frac{1}{r}\frac{\partial \Phi}{\partial \vartheta} - k\frac{\partial Z}{\partial r}, \tag{7.6}$$

$$\frac{h^3}{12}\psi = -D\frac{1}{r}\frac{\partial}{\partial \vartheta}(\Delta w) - \frac{\partial \Phi}{\partial r} - k\frac{1}{r}\frac{\partial Z}{\partial \vartheta}. \tag{7.7}$$

From (2.7.13) – (2.7.16) because of (6.32), and using (7.3) – (7.7), the internal moment resultants and transverse shear resultants can be written:

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2}+\nu\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \vartheta^2}\right)\right] + \\ +\frac{2D}{\delta^2}\left(\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}\right)\Delta w + \frac{2}{\delta^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \vartheta}\right) - \\ -k\left[Z - \frac{2}{\delta^2}\left(\frac{1}{r}\frac{\partial Z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 Z}{\partial \vartheta^2}\right)\right], \tag{7.8}$$

$$M_\vartheta = -D\left[\nu\frac{\partial^2 w}{\partial r^2}+\left(\frac{1}{r}\frac{\partial w}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w}{\partial \vartheta^2}\right)\right] + \frac{2D}{\delta^2}\frac{\partial^2}{\partial r^2}(\Delta w) - \\ -\frac{2}{\delta^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \vartheta}\right) - k\left(Z - \frac{2}{k^2}\frac{\partial^2 Z}{\partial r^2}\right), \tag{7.9}$$

$$H = -D(1-\nu)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \vartheta}\right) - \frac{2D}{\delta^2}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial \vartheta}(\Delta w)\right] + \\ +\Phi - \frac{2}{\delta^2}\frac{\partial^2 \Phi}{\partial r^2} - \frac{2k}{\delta^2}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial Z}{\partial \vartheta}\right), \tag{7.10}$$

$$N_r = -D\frac{\partial}{\partial r}(\Delta w) - k\frac{\partial Z}{\partial r}+\frac{1}{r}\frac{\partial \Phi}{\partial \vartheta}, \tag{7.11}$$

$$N_\vartheta = -D\frac{1}{r}\frac{\partial}{\partial \vartheta}(\Delta w) - k\frac{1}{r}\frac{\partial Z}{\partial \vartheta} - \frac{\partial \Phi}{\partial r}. \tag{7.12}$$

1. Bending of a plate in the shape of a circular sector subjected to a uniformly distributed load. Let a plate in the shape of a circular sector with an aperture angle θ_0 and radius a be subjected to a uniformly distributed normal load of intensity q (Figure 17).

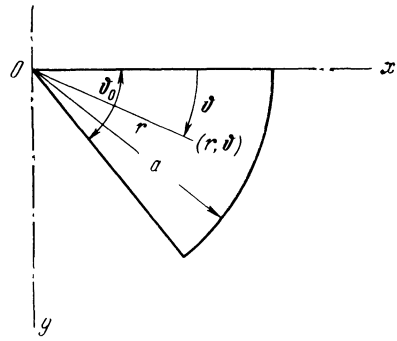

Figure 17.

Let the plate be simply-supported along the straight edges $\theta = 0$ and $\theta = \theta_0$ and be completely free along the arc r = a.

The boundary conditions of this problem are given, from (2.3.13) and (2.3.14), as:

at $\theta = 0$ and $\theta = \theta_0$

$$w = 0, \quad M_\vartheta = 0, \quad \varphi = 0; \tag{7.13}$$

at r = a

$$M_r = 0, \quad N_r = 0, \quad H = 0. \tag{7.14}$$

The uniformly distributed load of intensity q can be expanded into the series:

$$q = \sum_{n=1}^{\infty} q_n \sin \frac{n\pi\vartheta}{\vartheta_0}, \tag{7.15}$$

where the coefficients q_n are given as:

$$q_n = \begin{cases} 0 & (n = 2, 4, \ldots), \\ \dfrac{4q}{n\pi} & (n = 1, 3, \ldots). \end{cases} \tag{7.16}$$

The solution of the system (7.1) and (7.2) is sought in the form

$$w(r, \vartheta) = \sum_{n=1}^{\infty} W_n(r) \sin \frac{n\pi\vartheta}{\vartheta_0}, \tag{7.17}$$

$$\Phi(r, \vartheta) = \sum_{n=0}^{\infty} \Phi_n(r) \cos \frac{n\pi\vartheta}{\vartheta_0}, \tag{7.18}$$

It is evident that the boundary conditions (7.13) are satisfied identically.

Substituting values of Z = q, w and Φ from (7.15), (7.17) and (7.18), respectively, into (7.1) and (7.2) the following equations are obtained which are needed in order to determine the functions $W_n(r)$ and $\Phi_n(r)$:

$$\sum_{n=1}^{\infty}\left[\Delta_n\Delta_n W_n - \frac{q_n}{D}\right]\sin\lambda_n\vartheta = 0, \tag{7.19}$$

$$\sum_{n=0}^{\infty}[\Delta_n\Phi_n - \delta^2\Phi_n]\cos\lambda_n\vartheta = 0, \tag{7.20}$$

where

$$\Delta_n = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{\lambda_n^2}{r^2}, \qquad \lambda_n = \frac{n\pi}{\vartheta_0}. \tag{7.21}$$

On the basis of equations (7.19) and (7.20) and equating the expression in square brackets to zero we obtain ordinary differential equations which have solutions of the form

$$W_n(r) = c_{n1}r^{\lambda_n} + c_{n2}r^{\lambda_n+2} + c_{n3}r^{-\lambda_n} + c_{n4}r^{-\lambda_n-2} + \frac{q_n r^4}{D(\lambda_n^2 - 16)(\lambda_n^2 - 4)} \quad (n = 1, 2, \ldots), \tag{7.22}$$

$$\Phi_n(r) = c_{n5}I_{\lambda_n}(\delta r) + c_{n6}K_{\lambda_n}(\delta r) \qquad (n = 0, 1, 2, \ldots), \tag{7.23}$$

where c_{ni} are integration constants which are found from the boundary conditions; $I_p(x)$, $K_p(x)$ are Bessel functions of imaginary argument of the first and second kind [33].

Since we are considering a plate supported at the center (r = 0), then for any value of n it is necessary that we have:

$$c_{n3} = c_{n4} = c_{n6} = 0. \tag{7.24}$$

All remaining intergration constants are determined from the boundary conditions (7.14).

Substituting the expression for w and Φ from (7.17) and (7.18) into (7.8), (7.10), and (7.11) and considering the solutions (7.22) and (7.23), expressions for M_r, N_r and H are obtained in terms of the integration constants c_{n1}, c_{n2} and c_{n5}. By requiring that these values of M_r, N_r and H be equal to zero at r = a, an algebraic system of three equations with respect to the three integration constants is derived. When this system is solved with respect to c_{n1}, c_{n2} and c_{n5} we find that these constants become zero when n = 0, 2, 4, . . . while they are determined from the following formulas when n = 1, 3, 5 . . .

$$c_{n1} = -\frac{q_n a^{4-\lambda_n}}{2(1-\nu)D\lambda_n^2(\lambda_n+1)P_n}\left\{\frac{P_n}{\lambda_n-1}\left[2h^*\lambda_n + \right.\right.$$

$$+\frac{(\lambda_n^3+9\lambda_n^2+4\lambda_n+16)-\nu(\lambda_n^3+11\lambda_n^2+4\lambda_n-16)}{(\lambda_n+2)(\lambda_n^2-16)}\Big]-$$

$$-[(1-\nu)\lambda_n-2(1+\nu)]\left[\frac{2+\nu(\lambda_n+2)}{(\lambda_n+2)(\lambda_n+4)}+\right.$$

$$\left.\left.+\frac{2\lambda_n(\lambda_n+1)}{t^2(\lambda_n+2)}-h^*\lambda_n\right]\right\},$$

$$c_{n2} = \frac{q_n a^{2-\lambda_n}}{2D\lambda_n(\lambda_n+1)P_n}\left[\frac{2+\nu(\lambda_n+2)}{(\lambda_n+2)(\lambda_n+4)}+\frac{P_n}{\lambda_n^2-4}+\right.$$

$$\left.+\frac{2\lambda_n(\lambda_n+1)}{t^2(\lambda_n+2)}-h^*\lambda_n\right],$$

$$c_{n5} = -\frac{2q_n a^2}{\lambda_n P_n I_{\lambda_n}(t)}\left[\frac{2+\nu(\lambda_n+2)}{(\lambda_n+2)(\lambda_n+4)}-h^*\lambda_n+\frac{2\lambda_n(\lambda_n+1)}{t^2(\lambda_n+2)}\right];$$

The following notations are used here:

$$h^* = \frac{k}{a^2} = \left(2\frac{G}{G'}-\nu'\frac{E}{E'}\right)\frac{h^2}{10(1-\nu)a^2}, \quad t = \delta a = \frac{a}{h}\sqrt{\frac{10G'}{G}},$$

$$P_n = 3+\nu+\frac{8\lambda_n(\lambda_n+1)}{t^2}-\frac{4}{t}(\lambda_n+1)\frac{I_{\lambda_n-1}(t)}{I_{\lambda_n}(t)}.$$

Substituting values of c_{n1}, c_{n2} and c_{n5} into (7.22) and (7.23) and considering (7.24) as well as (7.17) – (7.19) the following expressions are obtained for the functions $w(r, \theta)$ and $\Phi(r, \theta)$:

$$w = \frac{2qa^4}{(1-\nu)D\vartheta_0}\sum_{n=1,3,\ldots}^{\infty}\left[(1-\nu)\lambda_n\left[\frac{2+\nu(\lambda_n+2)}{(\lambda_n+2)(\lambda_n+4)}-h^*\lambda_n+\right.\right.$$

$$\left.+\frac{2\lambda_n(\lambda_n+1)}{t^2(\lambda_n+2)}+\frac{P_n}{\lambda_n^2-4}\right]\left(\frac{r}{a}\right)^{\lambda_n+2}-$$

$$-\left\{[(1-\nu)\lambda_n-2(1+\nu)]\left[\frac{2+\nu(\lambda_n+2)}{(\lambda_n+2)(\lambda_n+4)}+\frac{2\lambda_n(\lambda_n+1)}{t^2(\lambda_n+2)}-h^*\lambda_n\right]+\right.$$

$$+\frac{P_n}{\lambda_n+1}\left[\frac{(\lambda_n^3+9\lambda_n^2+4\lambda_n+16)-\nu(\lambda_n^3+11\lambda_n^2+4\lambda_n-16)}{(\lambda_n+2)(\lambda_n^2-16)}+\right.$$

$$\left.\left.+2h^*\lambda_n\right]\right\}\left(\frac{r}{a}\right)^{\lambda_n}+\left.\frac{2(1-\nu)\lambda_n^2(\lambda_n+1)P_n}{(\lambda_n^2-4)(\lambda_n^2-16)}\left(\frac{r}{a}\right)^4\right]\frac{\sin\lambda_n\vartheta}{\lambda_n^3(\lambda_n+1)P_n} \tag{7.25}$$

$$\Phi = -\frac{8qa^2}{\vartheta_0} \sum_{n=1,3,\ldots}^{\infty} \left[\frac{2+\nu(\lambda_n+2)}{(\lambda_n+2)(\lambda_n+4)} + \frac{2\lambda_n(\lambda_n+1)}{t^2(\lambda_n+2)} - h^*\lambda_n \right] \frac{I_{\lambda_n}\left(t\frac{r}{a}\right)\cos\lambda_n\vartheta}{\lambda_n^2 P_n I_{\lambda_n}(t)}. \quad (7.26)$$

With the available expressions for w and Φ all other pertinent quantities can be determined. Final expressions are given below for only the maximum deflection and the bending moment M_θ at the point ($\theta = \theta_0/2$, r = a):

$$w_{\max} = \frac{4qa^4}{(1-\nu)D\vartheta_0} \sum_{n=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{\lambda_n^3(\lambda_n+1)P_n} \left\{ (1-\nu)\left[\frac{2\lambda_n(\lambda_n+1)}{t^2(\lambda_n+2)} + \frac{2+\nu(\lambda_n+2)}{(\lambda_n+2)(\lambda_n+4)} - h^*\lambda_n \right] + \frac{P_n}{\lambda_n - 1}\left[\frac{\lambda_n^2+2\lambda_n+2}{(\lambda_n+2)(\lambda_n+4)} - \frac{\nu(\lambda_n-1)}{(\lambda_n+4)} - h^*\lambda_n \right] \right\}, \quad (7.27)$$

$$M_{\vartheta\max} = \frac{4qa^2}{\vartheta_0} \sum_{n=1,3,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{\lambda_n^2 P_n} \left\{ P_n \left[\frac{1+\nu}{\lambda_n+2} + 2\left(\frac{1}{t^2} - h^*\right)\lambda_n \right] + 2(1+\nu)h^*\lambda_n - \frac{4(1+\nu)^2+2\nu\lambda_n}{(\lambda_n+2)(\lambda_n+4)} - \frac{4(1+\nu)\lambda_n(\lambda_n+1)}{t^2(\lambda_n+2)} \right\} \quad (7.28)$$

We see from equation (7.25) that the first term of the series at $\theta_0 = \pi/4$ ($\lambda_n = 4n$) and at $\theta_0 = \pi/2$ ($\lambda_n = 2n$) and the second term at $\theta_0 = 3\pi/4$ ($\lambda_n = 4n/3$) contain indeterminacies which can be easily found.

Finally, corresponding expressions and formulas are obtained for this problem similar to those found from the classical theory, by assuming that $h^* = 0$ and passing to the limit $t \to \infty$ [34].

2. Axisymmetric bending of a circular plate subjected to a self-equilibrating normally applied load. Let a circular transversely isotropic plate of radius a and thickness h with a free contour be bent by an axisymmetric self-equilibrating load of the type

$$Z = q\cos\alpha r. \quad (7.29)$$

The load Z is self-equilibrating if α satisfies the equation

$$\alpha a \sin\alpha a + \cos\alpha a = 1. \quad (7.30)$$

Due to the symmetry of the problem under consideration the system of initial equations (7.1) and (7.2) with consideration of (7.29) can be presented as follows

$$D\Delta_0\Delta_0 w = q\left[(1+k\alpha^2)\cos\alpha r + k\alpha\frac{\sin\alpha r}{r}\right], \tag{7.31}$$

$$\Delta_0\Phi - \delta^2\Phi = 0, \tag{7.32}$$

where

$$\Delta_0 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}. \tag{7.33}$$

Expressions for internal stress and moment resultants are also considerably simplified. From (7.8) – (7.12) these are:

$$M_r = D(1-\nu)\frac{1}{r}\frac{dw}{dr} - \left(1 - \frac{2}{\delta^2}\frac{1}{r}\frac{d}{dr}\right)(D\Delta_0 w + kq\cos\alpha r), \tag{7.34}$$

$$M_\vartheta = D(1-\nu)\frac{d^2w}{dr^2} - \left(1 - \frac{2}{\delta^2}\frac{d^2}{dr^2}\right)(D\Delta_0 w + kq\cos\alpha r), \tag{7.35}$$

$$H = \Phi - \frac{2}{\delta^2}\frac{d^2\Phi}{dr^2}, \tag{7.36}$$

$$N_r = -\frac{d}{dr}(D\Delta_0 w + kq\cos\alpha r), \quad N_\vartheta = -\frac{d\Phi}{dr}. \tag{7.37}$$

A general solution of equations (7.31) and (7.32) is

$$w = c_1 + c_2 r^2 + c_3 \ln\alpha r + c_4 r^2 \ln\alpha r +$$

$$+ \frac{q}{4D\alpha^4}[(5+4k\alpha^2)\cos\alpha r - \alpha r\sin\alpha r + (6+4k\alpha^2-\alpha^2r^2)R(r)], \tag{7.38}$$

$$\Phi = c_5 I_0(\delta r) + c_6 K_0(\delta r), \tag{7.39}$$

where $c_1, \ldots, c_6$ are integration constants; $I_0(x)$, $K_0(x)$ are Bessel functions of imaginary argument of the first and second kind of zero order, and

$$R(r) = \int \frac{1-\cos\alpha r}{r}\,dr.$$

By excluding a rigid body displacement and considering that the stress and moment resultants of the plate should be finite at $r \to 0$, we assume that

$$c_1 = 0, \qquad c_3 = 0, \qquad c_6 = 0. \tag{7.40}$$

The remaining integration constants, c_2, c_4 and c_5, are determined from the boundary conditions at the free plate edge which because of (2.3.13) are

at r = a

$$M_r = 0, \qquad N_r = 0, \qquad H = 0. \tag{7.41}$$

Satisfying the boundary conditions (7.41) we obtain the integration constants:

$$\left.\begin{aligned} c_2 &= \frac{q}{8(1+\nu)D\alpha^2}[3+\nu+4\cos\alpha a+2(1+\nu)R(a)], \\ c_4 &= 0, \qquad c_5 = 0 \end{aligned}\right\} \tag{7.42}$$

Substituting expressions (7.40) and (7.42) into (7.38) and (7.39) we obtain the functions w and Φ:

$$w = \frac{q}{4D\alpha^4}\left\{\frac{\alpha^2 r^2}{2(1+\nu)}[3+\nu+4\cos\alpha a+2(1+\nu)R(a)]+\right.$$
$$\left.+(5+4k\alpha^2)\cos\alpha r-\alpha r\sin\alpha r+(6+4k\alpha^2-\alpha^2r^2)R(r)\right\}, \qquad \Phi\equiv 0. \tag{7.43}$$

With these available expressions for w and Φ it is easy to determine all other necessary quantities.

In the particular case of an isotropic plate the parameter k is obtained from (7.4):

$$k = \frac{(2-\nu)h^2}{10(1-\nu)}. \tag{7.44}$$

Substituting this value of k into (7.43) the normal displacement w can be written:

$$w = \frac{q}{4D\alpha^4}\left\{\frac{\alpha^2r^2}{2(1+\nu)}[3+\nu+4\cos\alpha a+2(1+\nu)R(a)]+\right.$$
$$+5\cos\alpha r-\alpha r\sin\alpha r+(6-\alpha^2r^2)R(r)+$$
$$\left.+(8-4\nu)\frac{\alpha^2h^2}{10(1-\nu)}[\cos\alpha r+R(r)]\right., \tag{7.45}$$

where the last term with multiplier (8-4ν) represents a correction to the classical theory.

This problem for the isotropic plate was solved before [35] with the use of another more consistent theory [36]. A comparison of our results with those of A. V. Kolos (after correcting certain inaccuracies in his work) shows that they differ only in the terms which characterize the correction to the classical theory. Furthermore, this difference is only in

the multiplier of the correction term, namely, instead of (8-4 ν) the above work obtains (8-3 ν). This difference is insignificant and disappears altogether at $\nu = 0$.

8. Large Deflections of Long Orthotropic Plates into a Cylindrical Surface

Consider a long rectangular orthotropic plate loaded with a transverse uniformly distributed load applied at the external plate surfaces in the following manner:

$$Z^+ = q, \qquad Z^- = 0, \tag{8.1}$$

On the basis of this and according to (2.2.29)

$$Z_1 = \frac{q}{2}, \qquad Z_2 = q. \tag{8.2}$$

Furthermore, let the plate be uniformly supported along the long sides and arbitrarily supported along the short sides. The bent surface of the plate at a distance from the short sides will resemble a cylindrical surface. By placing the origin of coordinates on one of the long sides away from the plate short sides and directing the x-axis along the long side, we consider that the undetermined quantities of the problem, w, φ, ψ, and F, are functions of the coordinate y only. (Figure 18)

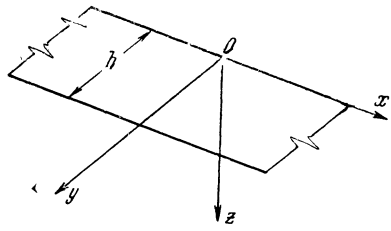

Figure 18.

The functions w(y), φ(y), ψ(y) and F(y) are determined from the system of differential equations (2.8.29) – (2.8.32) which in this case because of (2.8.27) and (8.2) is

$$\frac{h^3}{12}\frac{d\psi}{dy} + T_y\frac{d^2w}{dy^2} = -q, \tag{8.3}$$

$$\frac{h^2}{10}a_{55}D_{66}\frac{d^2\varphi}{dy^2} - \frac{h^3}{12}\varphi = 0, \tag{8.4}$$

$$D_{22}\frac{d^3w}{dy^3} - \frac{h^2}{10}a_{44}D_{22}\frac{d^2\psi}{dy^2} + \frac{h^3}{12}\psi - \frac{h^5}{120}A_2\frac{d^2\psi}{dy^2} = 0. \tag{8.5}$$

Without mentioning details it should be indicated, however, that for a very long plate (with respect to x) it has been assumed that the force T_y is constant

for the entire plate area under consideration and the strain ϵ_x is zero. Because of these facts equation (2.8.12) and therefore (2.8.32) are not considered [37, 38].

From (8.3) and letting $T_y = T$ (here and below the subscript y on T_y is omitted)

$$\psi = -\frac{12}{h^3} yq - \frac{12}{h^3} T \frac{dw}{dy} + c_1, \tag{8.6}$$

where c_1 is the integration constant.

From the second equation of the initial system, i.e., from (8.4), we have for $\varphi(y)$:

$$\varphi(y) = c_2 e^{ky} + c_3 e^{-ky}, \tag{8.7}$$

where c_2, c_3 are integration constants, and k is defined such that

$$k^2 = \frac{5h}{6a_{55}D_{66}}.$$

Finally, from (8.5) with the use of (8.3) to eliminate $\psi(y)$, we obtain the differential equation for w(y):

$$\frac{d^4w}{dy^4} - \frac{T}{D}\frac{d^2w}{dy^2} = \frac{q}{D}, \tag{8.8}$$

where

$$D = D_{22}\left[1 + \left(\frac{6}{5h} a_{44} + \frac{h^2}{10}\frac{A_2}{D_{22}}\right) T\right]. \tag{8.9}$$

Thus, we arrived at the differential equation (8.8) which is solved identically for given boundary conditions at the long sides of the plate (y = 0, y = b).

A case is discussed below where the effect of the normal stress σ_z is disregarded, i.e., in all expressions $A_1 = A_2 = 0$.

By introducing dimensionless parameters

$$m^* = \frac{6B_{22}h^2}{5B_{44}b^2}, \quad T^* = \frac{Tb^2}{B_{22}h^3}, \tag{8.10}$$

the differential equation (8.8) can be rewritten in the form

$$\frac{d^4w}{dy^4} - \lambda^2 \frac{d^2w}{dy^2} = \frac{\lambda^2}{T} q, \tag{8.11}$$

where

$$\lambda^2 = \frac{12T^*}{b^2(1 + m^*T^*)}. \tag{8.12}$$

A general solution of equation (8.11) using the conventional notation is

$$w = c_4 \operatorname{sh} \lambda y + c_5 \operatorname{ch} \lambda y + c_6 y + c_7 - \frac{qy^2}{2T}, \tag{8.13}$$

where $c_4, \ldots, c_7$ are integration constants, and the last term represents a particular solution corresponding to the right-hand side of equation (8.11).

The integration constants $c_1, \ldots, c_7$ are determined, as a rule, from the boundary conditions. However, considering that the determination of the relationships between q, T, w, $\sigma_{y,\text{deformed.}}$, is of interest, only the integration constants present in (8.13) are determined. All remaining integration constants are determined in the same way as described in Section 1 of this chapter.

Two boundary conditions along the long sides of the plate are considered below.

1. Plate simply-supported along the long sides. The boundary conditions imposed on w(y) in the case of simple supports are presented in the following manner on the basis of (2.3.14), (2.8.21) and (8.6):

$$\left.\begin{array}{ll} \text{at } y = 0 & \\ & w = 0, \quad \dfrac{d^2w}{dy^2} + \dfrac{\lambda^2}{T}\left(\dfrac{1}{\lambda^2} - \dfrac{D_{22}}{T}\right) q = 0; \\ \text{at } y = b & \\ & w = 0, \quad \dfrac{d^2w}{dy^2} + \dfrac{\lambda^2}{T}\left(\dfrac{1}{\lambda^2} - \dfrac{D_{22}}{T}\right) q = 0. \end{array}\right\} \tag{8.14}$$

After the integration constants $c_4, \ldots, c_7$ are determined from the boundary conditions (8.14) and equation (8.13) we have a solution for the normal plate displacement which is obtained by substituting their values back into (8.13):

$$w = q\frac{D_{22}}{T^2}\left[\frac{\operatorname{ch} \lambda\left(\dfrac{b}{2} - y\right)}{\operatorname{ch}\dfrac{\lambda b}{2}} - 1\right] - q\frac{y(y-b)}{2T} \tag{8.15}$$

The line of maximum deflection will be at y = b/2 and this deflection is determined to be

$$f = q\frac{D_{22}}{T^2}\left(\frac{1}{\operatorname{ch}\dfrac{\lambda b}{2}} - 1\right) + q\frac{b^2}{8T}. \tag{8.16}$$

The force T is easily found by considering the stationary conditions of the plate supports. The total relative displacement of the supports, Δ, is:

$$\Delta = -\int_0^b \frac{dv}{dy}\,dy. \tag{8.17}$$

From (2.8.18) and considering (2.8.13) and (2.8.27) we obtain for dv/dy:

$$\frac{dv}{dy} = \frac{T}{C_{22}} - \frac{1}{2}\left(\frac{dw}{dy}\right)^2 \tag{8.18}$$

Notice that (8.18) can be obtained directly from (2.8.34) if we consider that a_{22}/h is equal to $1/C_{22}$ (along the x direction) for the long strip under study.

Substituting the expression for dv/dy from (8.18) into (8.17) and considering (8.15) we obtain an expression with which to determine T:

$$T\frac{b}{C_{22}} = q^2\frac{D_{22}^2\lambda^2 b}{T^2}\left(\frac{\operatorname{th}\frac{\lambda b}{2}}{2\lambda b} - \frac{1}{4\operatorname{ch}^2\frac{\lambda b}{2}} - \right.$$

$$\left. - \frac{T}{D_{22}\lambda^2} + \frac{2T}{D_{22}\lambda^3 b}\operatorname{th}\frac{\lambda b}{2} + \frac{T^2 b^2}{24 D_{22}^2\lambda^2}\right) \tag{8.19}$$

Finally, from (2.8.15) because of (8.6) and (8.15) the extremal bending stress ($z = \pm h/2$, $y = b/2$)is found to be:

$$\overset{\max}{\underset{\min}{\sigma_{\text{bending}}}} = \pm hB_{22}\left[\left(1 + \frac{a_{44}}{h}T\right)\left(\frac{D_{22}\lambda^2 q}{T^2\operatorname{ch}\frac{\lambda b}{2}} - \frac{q}{T}\right) + \frac{a_{44}}{h}q\right] \tag{8.20}$$

By introducing dimensionless quantities for the load (q*), stress (σ^*), and maximum deflection (ζ^*)

$$q^* = \frac{qb^4}{B_{22}h^4}, \quad \sigma^* = \frac{\sigma_{\text{bending}}b^2}{B_{22}h^2}, \quad \zeta^* = \frac{f}{h}, \tag{8.21}$$

We obtain from (8.15), (8.19) and (8.20) the final equations of the problem:

$$\zeta^* = \frac{q^*}{8T^*}\left[1 + \frac{2}{3}\frac{1}{T^*}\left(\frac{1}{\operatorname{ch}\frac{\lambda b}{2}} - 1\right)\right], \tag{8.22}$$

$$T^* = \frac{\lambda^2 b^2 (q^*)^2}{144 (T^*)^2}\left(\frac{\operatorname{th}\frac{\lambda b}{2}}{2\lambda b} - \frac{1}{4\operatorname{ch}^2\frac{\lambda b}{2}} - \frac{12T^*}{\lambda^2 b^2} + \frac{24T^*}{b^2\lambda^3}\operatorname{th}\frac{\lambda b}{2} + \frac{6(T^*)^2}{\lambda^2 b^2}\right), \tag{8.23}$$

$$\sigma^* = \pm\frac{q^*}{T^*}\left[\left(1 + \frac{5}{6}m^*T^*\right)\left(\frac{1}{(1 + m^*T^*)\operatorname{ch}\frac{\lambda b}{2}} - 1\right) + \frac{5}{6}m^*T^*\right]. \tag{8.24}$$

Let us study now the effect of transverse shear on the deflection of the centerline and on the bending stresses. For this purpose tables have been compiled at different values of m* which characterize the influence of transverse shear and which show the relationships between q*, ζ* and σ* for this problem.

Because of the transcendental nature of equation (8.23) values of q* are determined for given values of T*. Using the corresponding values of T* and q* and also formulas (8.22) and (8.24), the corresponding maximum deflections (ζ*) and maximum bending stresses (σ*) are determined.

	T^*	0,2	0,4	1,0	3,0
$m^* = 0$	q^*	2,275	3,827	8,991	32,61
	ζ^*	0,284	0,400	0,631	1,086
	σ^*	1,362	1,902	2.953	4,894
$m^* = 0{,}5$	q^*	1,724	3,064	7,795	30,49
	ζ^*	0.280	0,395	0,623	1,066
	σ^*	1.004	1,461	2 209	3,753

In this table calculations for two values of m* are given. The first case of m* = 0 represents the classical theory which is indifferent toward B_{22}/B_{44}.

It is evident from this table that the maximum deflection ζ* and the stress resultant at the midplane T* increase, for a constant load, with an

increase in the shear coefficient, i.e., with a decrease of $B_{44} = G_{23}$, when transverse shears are considered. The classical theory produces too small values for ζ^* and T^* at the same loads. In the case of the maximum bending stresses, σ^*, the classical theory produces values that are too high.

2. Plate clamped along the long sides. Boundary conditions imposed on w(y) in the case of rigid clamping are obtained from (2.3.15) on the assumption that $z_0 = 0$, i.e., on the assumption that the vertical elements of the plate adjacent to the midplane are fixed:

$$\left.\begin{array}{ll} \text{at } y = 0 & \\ & w = 0, \quad -\dfrac{dw}{dy} + a_{44}\dfrac{h^2}{8}\psi = 0, \\ \text{at } y = b & \\ & w = 0, \quad -\dfrac{dw}{dy} + a_{44}\dfrac{h^2}{8}\psi = 0. \end{array}\right\} \tag{8.25}$$

The boundary conditions (8.25) will be transformed somewhat, and the undetermined quantity $\psi(y)$ will be eliminated from them. Due to the absence of concentrated forces distributed along the line $y = b/2$ and also due to the symmetrical shape of the plate bending with respect to the line $y = b/2$ we have

at $y = \frac{b}{2}$

$$\psi = 0, \quad \frac{dw}{dy} = 0. \tag{8.26}$$

Using conditions (8.26) we obtain for $\psi(y)$, from (8.6):

$$\psi = \frac{12}{h^3}\left[\left(\frac{b}{2} - y\right)q - \frac{dw}{dy}T\right] \tag{8.27}$$

Substituting this expression for ψ from (8.27) into (8.25), the boundary conditions become

$$\left.\begin{array}{ll} \text{at } y = 0 & \\ & w = 0, \quad -\dfrac{dw}{dy} + \dfrac{3a_{44}\left(\dfrac{b}{2} - y\right)}{2h + 3a_{44}T}q = 0; \\ \text{at } y = b & \\ & w = 0, \quad -\dfrac{dw}{dy} + \dfrac{3a_{44}\left(\dfrac{b}{2} - y\right)}{2h + 3a_{44}T}q = 0. \end{array}\right\} \tag{8.28}$$

From conditions (8.28) with the use of (8.13) the integration constants $c_4, \ldots, c_7$ are determined. Substitution of these results into (8.13) produces the solution for w(y):

$$w = q\frac{b\left(1-\dfrac{3a_{44}T}{2h+3a_{44}T}\right)}{2T\lambda\,\mathrm{th}\dfrac{\lambda b}{2}}\left[\frac{\mathrm{ch}\,\lambda\left(\dfrac{b}{2}-y\right)}{\mathrm{ch}\dfrac{\lambda b}{2}}-1\right]+q\frac{y(b-y)}{2T}. \tag{8.29}$$

Next, using the assumption of immovable supports ($\Delta = 0$) and by repeating the method and logical steps of the previous problem, the following equations are obtained, which determine the relation between the dimensionless load q*, dimensionless stress resultant of the middle surface T*, dimensionless maximum deflection ζ^* and dimensionless bending stress σ^* (the bending stresses correspond to points (y = 0, z = ±h/2) and (y = b, z = ±h/2) where they are extremal):

$$\zeta^* = \frac{0{,}4q^*}{(m^*T^*+0{,}8)\lambda b T^*\,\mathrm{th}\dfrac{\lambda b}{2}}\left(\frac{1}{\mathrm{ch}\dfrac{\lambda b}{2}}-1\right)+\frac{q^*}{8T^*}, \tag{8.30}$$

$$T^* = \frac{(q^*)^2}{(T^*)^2}\frac{0{,}4}{m^*T^*+0{,}8}\left[\frac{2}{\lambda^2 b^2}+\frac{m^*T^*+0{,}8}{9{,}6}-\right.$$
$$\left.-\frac{0{,}1}{(m^*T^*+0{,}8)\,\mathrm{sh}^2\dfrac{\lambda b}{2}}-\frac{m^*T^*+0{,}6}{(m^*T^*+0{,}8)\lambda b\,\mathrm{th}\dfrac{\lambda b}{2}}\right], \tag{8.31}$$

$$\sigma^* = \pm\frac{q^*}{2T^*}\left[\left(1+\frac{5}{6}m^*T^*\right)\left(\frac{0{,}4b\lambda}{(m^*T^*+0{,}8)\,\mathrm{th}\dfrac{\lambda b}{2}}-1\right)+\frac{5}{6}m^*T^*\right] \tag{8.32}$$

Just as before, some numerical results are given below to show the influence of transverse shear on the pertinent quantities of the problem. The bending stresses are determined for z = h/2.

	T^*	1,0	3,0
$m^* = 0$	q^*	25,82	67,42
	ζ^*	0,62	1,11
	σ^*	10,90	22,64
$m^* = 0,5$	q^*	10,77	35,58
	ζ^*	0,63	1,08
	σ^*	2,08	3,27

A comparison of the results obtained shows that they resemble the case of a simply-supported plate. Namely, the classical theory ($m^* = 0$) for the maximum deflection ζ^* and for mid-plane stress resultants T^* produces values that are too low and values for the maximum bending stresses σ^* which are too high. However, the quantitative difference in results between the classical and improved theories for the case of rigidly clamped edges is considerably greater than the difference for the case of simply-supported edges. This indicates that the nature of the boundary conditions also influences the accuracy of results obtained with the classical theory.

9. Large Deflections of Simply-Supported Orthotropic Plates

Consider the transverse bending of a simply-supported rectangular (a x b) orthotropic plate loaded by a uniformly distributed load of intensity

$$Z_1 = \frac{q}{2}, \quad Z_2 = q. \tag{9.1}$$

It is assumed that the ratio of the sides a/b is within $1 \leqslant a/b \leqslant 2$, and the moduli of elasticity E_x and E_y, generally speaking, differ very little; otherwise, the approximate solution accepted in this section would not be acceptable [39, 40] (Figure 13).

This problem can be reduced to the solution of the system of equations (2.8.29) – (2.8.32) for the load (9.1). The applicable boundary conditions are:

$$\left.\begin{aligned}
&\text{at } x = 0,\ x = a\\
&\quad w = 0, \quad M_x = 0, \quad \psi = 0, \quad T_x^0 = 0, \quad S = 0,\\
&\text{at } y = 0,\ y = b\\
&\quad w = 0, \quad M_y = 0, \quad \varphi = 0, \quad T_y^0 = 0, \quad S = 0,
\end{aligned}\right\} \tag{9.2}$$

where

$$T_x^0 = \frac{1}{b}\int_0^b T_x dy, \qquad T_y^0 = \frac{1}{a}\int_0^a T_y dx$$

are the average values of the inplane stress resultants at the corresponding plate edges.

The first three conditions for simple supports will be satisfied along all plate edges if we assume for w, φ and ψ

$$\left.\begin{aligned}
w &= f \sin\frac{\pi x}{a}\sin\frac{\pi y}{b},\\
\varphi &= c_\varphi \cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\\
\psi &= c_\psi \sin\frac{\pi x}{a}\cos\frac{\pi y}{b}.
\end{aligned}\right\} \tag{9.3}$$

Substituting these expressions for w, φ and ψ from (9.3) into (2.8.30) and (2.8.31) and solving them with respect to the coefficients c_φ and c_ψ we obtain

$$c_\varphi = \frac{\Delta_\varphi}{\Delta} f, \qquad c_\psi = \frac{\Delta_\psi}{\Delta} f, \tag{9.4}$$

where

$$\Delta = \frac{h^6}{144}(1 - \lambda_0), \tag{9.5}$$

$$\left.\begin{aligned}
\Delta_\varphi &= \frac{h^3}{12}\left[D_{11}\frac{\pi^3}{a^3} + (D_{12} + 2D_{66})\frac{\pi^3}{ab^2}\right](1 + \lambda_1),\\
\Delta_\psi &= \frac{h^3}{12}\left[D_{22}\frac{\pi^3}{b^3} + (D_{12} + 2D_{66})\frac{\pi^3}{ba^2}\right](1 + \lambda_2),
\end{aligned}\right\} \tag{9.6}$$

$$\left.\begin{aligned}
\lambda_0 &= \frac{h^2}{10}\pi^2\left[\alpha_2\frac{1}{b^2}+B_{66}\frac{1}{a^2}\right]a_{44}+\frac{h^2}{10}\pi^2\left[\alpha_1\frac{1}{a^2}+B_{66}\frac{1}{b^2}\right]a_{55}+\\
&+\frac{h^4}{100}\pi^4\left\{\left[\alpha_1\frac{1}{a^2}+B_{66}\frac{1}{b^2}\right]\left[\alpha_2\frac{1}{b^2}+B_{66}\frac{1}{a^2}\right]-\beta_1\beta_2\frac{1}{b^2a^2}\right\}a_{44}a_{55},\\
\lambda_1 &= -\frac{h^2}{10}\pi^2\left[\frac{D_{22}/b^3+(D_{12}+2D_{66})/ba^2}{D_{11}/a^3+(D_{12}+2D_{66})/ab^2}\frac{\beta_1}{ab}-\right.\\
&\qquad\left.-\left(\alpha_2\frac{1}{b^2}+B_{66}\frac{1}{a^2}\right)\right]a_{44},\\
\lambda_2 &= -\frac{h^2}{10}\pi^2\left[\frac{D_{11}/a^3+(D_{12}+2D_{66})/ab^2}{D_{11}/a^3+(D_{12}+2D_{66})/ab^2}\frac{\beta_2}{ab}-\right.\\
&\qquad\left.-\left(\alpha_1\frac{1}{a^2}+B_{66}\frac{1}{b^2}\right)\right]a_{55},
\end{aligned}\right\} \tag{9.7}$$

$$\left.\begin{aligned}
\alpha_1 &= \left(B_{11}+\frac{A_1}{a_{55}}\right), \qquad \beta_1 = \left(B_{12}+B_{66}+\frac{A_1}{a_{44}}\right),\\
\alpha_2 &= \left(B_{22}+\frac{A_2}{a_{44}}\right), \qquad \beta_2 = \left(B_{12}+B_{66}+\frac{A_2}{a_{55}}\right).
\end{aligned}\right\} \tag{9.8}$$

It is evident that the λ_i represent additional terms which originate because of the consideration of transverse shear and the normal stress σ_z.

Thus the functions φ and ψ are now represented in terms of the undetermined parameter f. We will also attempt to represent the function F in terms of this same parameter.

Substituting the expression for w from (9.3) into (2.8.32) the following expression is obtained, after some preliminary manipulations:

$$\frac{a_{22}}{h}\frac{\partial^4F}{\partial x^4}+\frac{a_{66}+2a_{12}}{h}\frac{\partial^4F}{\partial x^2\partial y^2}+\frac{a_{11}}{h}\frac{\partial^4F}{\partial y^4}=\frac{\pi^4}{2a^2b^2}\left(\cos\frac{2\pi x}{a}+\cos\frac{2\pi y}{b}\right)f^2 \tag{9.9}$$

It was assumed during the derivation of (9.9) that q and consequently Z_1 are constant.

A general solution of (9.9), which satisfies boundary conditions (9.2) for T_x^0, T_y^0 and S, can be presented in the following form (see also (2.8.27)) [41]

$$F(x,y)=\frac{hf^2}{32}\left[\frac{1}{a_{22}}\left(\frac{a}{b}\right)^2\cos\frac{2\pi x}{a}+\frac{1}{a_{11}}\left(\frac{b}{a}\right)^2\cos\frac{2\pi y}{b}\right]. \tag{9.10}$$

Consequently, the function F(x, y) has also been represented in terms of the undetermined parameter f. In other words, all three equations of the principal system, namely, (2.8.30), (2.8.31) and (2.8.32), for the boundary conditions

(9.2), have been solved accurately and all of the desired functions are represented in terms of the parameter f. Only equation (2.8.29) remains unused, as well as the solution for the parameter f.

In order to solve the equation (2.8.29) we need the Bubnov-Galerkin equation

$$\int_0^a \int_0^b L(x, y) \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} dy\, dx = 0, \tag{9.11}$$

where

$$L(x, y) = \frac{h^3}{12}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\right) + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 w}{\partial x\, \partial y}\frac{\partial^2 F}{\partial x\, \partial y} + q. \tag{9.12}$$

Substituting expressions for $\varphi(x, y)$, $\psi(x, y)$, $F(x, y)$ and $w(x, y)$ into (9.11) and integrating leads us to an equation which relates the deflection coefficient f to the external load q

$$\left\{\left[D_{11}\frac{\pi^4}{a^4} + (D_{12} + 2D_{66})\frac{\pi^4}{a^2 b^2}\right]\left(1 - \frac{\lambda_0 - \lambda_1}{1 + \lambda_0}\right) + \left[D_{22}\frac{\pi^4}{b^4} + (D_{12} + 2D_{66})\frac{\pi^4}{a^2 b^2}\right]\left(1 - \frac{\lambda_0 - \lambda_2}{1 + \lambda_0}\right)\right\} f + \frac{\pi^4 h}{16 a^2 b^2}\left[\frac{1}{a_{22}}\left(\frac{a}{b}\right)^2 + \frac{1}{a_{11}}\left(\frac{b}{a}\right)^2\right] f^3 = \frac{16}{\pi^2} q. \tag{9.13}$$

Thus, the first approximation to the problem has been obtained. It is easy to find the functions ($\varphi(x, y)$, $\psi(x, y)$, $F(x, y)$ and $w(x, y)$) with the available value of f using equations (9.3), (9.4) and (9.10), for different values of q, as well as all other quantities of interest.

Following the usual procedures [42], it is possible to derive further approximations for the problem under consideration.

It is evident from equation (9.13) that corrections to the classical theory due to the inclusion of transverse shear and normal stress σ_z are represented by $(\lambda_0 - \lambda_i) / (1 + \lambda_0)$, which depends on the geometry as well as on the ratio of the elastic constants of the material.

In order to have an idea about the magnitude of the corrections present in (9.13) the tables below show numerical values of the quantity

$$\lambda^* = 1 - \frac{\lambda_0 - \lambda_i}{1 + \lambda_0}$$

for square (a = b) transversely isotropic plates with Poisson's ratios of $\nu = 0.3$, $\nu' = 0.25$.

$h/a = 1/5$

E/E′ \ E/G′	0,0	2,0	5,0	10,0
0,0	1,000	0,852	0,692	0,536
2,0	1,059	0,896	0,729	0,556

$h/a = 1/10$

E/E′ \ E/G′	0,0	2,0	5,0	10,0
0,0	1,000	0,958	0,902	0,828
2,0	1,014	0,972	0,914	0,832

These tables show that the correction λ^* for the classical theory ($E/G' = 0$, $E/E' = 0$) is equal to zero. The correction in the case of the improved theory, which is defined by inclusion of the effects of transverse shear ($E/G' \neq 0$) and of the normal stress σ_z ($E/E' \neq 0$), is not zero and may be large for certain ratios of the elastic constants of the plate material.

Particular theory. The problem of a simply-supported rectangular plate undergoing large deflections may be solved with particular theory [43].

Consider a transversely isotropic plate when the effect of transverse shear is considered. For this kind of plate on the basis of (1.4.10), (1.4.11), (2.5.1), (2.5.2), and (2.10) and using equations (2.8.65) and (2.8.66), the following system of governing equations is obtained:

$$\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = Eh\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right], \quad (9.14)$$

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2 F}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2}\right) + \frac{Q}{D}, \quad (9.15)$$

where

$$Q = q - \frac{ED}{(1-\nu^2)G'}\frac{h^2}{10}\left(\frac{\partial^6 w_0}{\partial x^6} + 3\frac{\partial^6 w_0}{\partial x^4 \partial y^2} + 3\frac{\partial^6 w_0}{\partial x^2 \partial y^4} + \frac{\partial^6 w_0}{\partial y^6}\right), \quad (9.16)$$

The normal displacement w_0 is found from the classical theory, i.e., from this system of equations:

$$\frac{\partial^4 F_0}{\partial x^4} + 2\frac{\partial^4 F_0}{\partial x^2 \partial y^2} + \frac{\partial^4 F_0}{\partial y^4} = Eh\left[\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} - \left(\frac{\partial^2 w_0}{\partial x\,\partial y}\right)^2\right], \quad (9.17)$$

$$\frac{\partial^4 w_0}{\partial x^4} + 2\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + \frac{\partial^4 w_0}{\partial y^4} =$$

$$= \frac{1}{D}\left(\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 F_0}{\partial y^2} - \frac{\partial^2 w_0}{\partial x\,\partial y}\frac{\partial^2 F_0}{\partial x\,\partial y} + \frac{\partial^2 w_0}{\partial y^2}\frac{\partial^2 F_0}{\partial x^2}\right) + \frac{q}{D}. \quad (9.18)$$

The boundary conditions of simple support, because of (9.2), (2.8.60), (2.8.62) and (2.8.52), are

$$\left.\begin{aligned} &\text{for } x = 0,\ x = a \\ &\qquad w = 0, \quad \frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial^2 F}{\partial y^2} = 0, \quad \frac{\partial^2 F}{\partial x\,\partial y} = 0; \\ &\text{for } y = 0,\ y = b \\ &\qquad w = 0, \quad \frac{\partial^2 w}{\partial y^2} = 0, \quad \frac{\partial^2 F}{\partial x^2} = 0, \quad \frac{\partial^2 F}{\partial x\,\partial y} = 0. \end{aligned}\right\} \quad (9.19)$$

Replacing w and F with w_0 and F_0 in (9.19) produces the boundary conditions of the classical theory.

The first step is to find $w_0(x, y)$, i.e., to obtain the solution of the problem in the classical form.

This may be accomplished by the method of independently selecting approximating functions [44] which has been successfully used for the solution of some problems of the bending of rectangular plates with the consideration of large deflections [45].

Without considering unnecessary details, a general procedure for obtaining the classical solution is given below. From equations (9.14), (9.15), (9.17), (9.18) as well as the boundary conditions (9.19) we see that the method of solution of the classical equations is also completely acceptable for the improved theory.

The solution of the system (9.17) and (9.18) is sought in the form

$$F_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{0mn} U_m(x)\, V_n(y), \quad (9.20)$$

$$w_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{0mn} X_m(x)\, Y_n(y), \quad (9.21)$$

where A_{0mn} and f_{0mn} are undetermined constants; $U_m(x)$, $V_n(y)$, $X_m(x)$, $Y_n(y)$ are functions which are selected in such a way that the boundary conditions (9.19) are satisfied.

Assuming that $U_m(x), \ldots, Y_n(y)$ are characteristic shapes found as solutions of the natural vibration of a beam, the functions are

$$F_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{0mn}\left[\sin\frac{\lambda_m x}{a} - \operatorname{sh}\frac{\lambda_m x}{a} - \right.$$
$$\left. - \alpha_m\left(\cos\frac{\lambda_m x}{a} - \operatorname{ch}\frac{\lambda_m x}{a}\right)\right]\left[\sin\frac{\lambda_n y}{b} - \right.$$
$$\left. - \operatorname{sh}\frac{\lambda_n y}{b} - \alpha_n\left(\cos\frac{\lambda_n y}{b} - \operatorname{ch}\frac{\lambda_n y}{b}\right)\right], \qquad (9.22)$$

$$w_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{0mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \qquad (9.23)$$

where

$$\alpha_j = (\operatorname{sh}\lambda_j - \sin\lambda_j) \;/\; (\operatorname{ch}\lambda_j - \cos\lambda_j) \;;$$

λ_j are the roots of the transcendental equation

$$\operatorname{ch}\lambda_j \cos\lambda_j = 1, \qquad \text{i.e., } \lambda_j = (j + 0.5\pi).$$

To determine the constants A_{0mn} and f_{0mn} we use the basic principles of the method of independently selecting the approximating functions, namely, equation (9.17) is multiplied by the variation of the function F, and equation (9.18) is multiplied by the variation of the function w. After integrating the obtained expression over the entire plate area the following system of two equations is obtained:

$$\int_0^a\int_0^b \Phi U_m(x)\,V_n(y)\,dx\,dy = 0, \qquad (9.24)$$

$$\int_0^a\int_0^b W X_m(x)\,Y_n(y)\,dx\,dy = 0, \qquad (9.25)$$

where

$$\Phi \equiv \Delta\Delta F_0 - Eh\left[\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} - \left(\frac{\partial^2 w_0}{\partial x\,\partial y}\right)^2\right] = 0,$$

$$W \equiv \Delta\Delta w_0 - \frac{1}{D}\left(q + \frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 F_0}{\partial y^2} - \frac{\partial^2 w_0}{\partial x\,\partial y}\frac{\partial^2 F_0}{\partial x\,\partial y} + \frac{\partial^2 w_0}{\partial y^2}\frac{\partial^2 F_0}{\partial x^2}\right) = 0,$$

$$\Delta\Delta = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^2}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}.$$

Using only the first approximation, i.e., by assuming that

$$F_0 = A_0 U(x)\,V(y), \qquad w_0 = f_0 X(x)\,Y(y), \tag{9.26}$$

where

$$U(x) = \sin\frac{\lambda x}{a} - \operatorname{sh}\frac{\lambda x}{a} - \alpha\left(\cos\frac{\lambda x}{a} - \operatorname{ch}\frac{\lambda x}{a}\right),$$

$$V(y) = \sin\frac{\lambda y}{b} - \operatorname{sh}\frac{\lambda y}{b} - \alpha\left(\cos\frac{\lambda y}{b} - \operatorname{ch}\frac{\lambda y}{b}\right),$$

$$X(x) = \sin\frac{\pi x}{a}, \qquad Y(y) = \sin\frac{\pi y}{b},$$

$$\alpha = \frac{\operatorname{sh}\lambda - \sin\lambda}{\operatorname{ch}\lambda - \cos\lambda} = 1{,}01781, \qquad \lambda = 1{,}50562\pi,$$

the boundary conditions (9.19) are completely satisfied and from (9.24) and (9.25) we obtain

$$A_0 = -\frac{33 f_0^2 Eh}{537{,}2\,(e^{-2} + e^2) + 322{,}5} \tag{9.27}$$

as well as an equation which establishes the relationship between the bending load q(x, y) and the deflection at the plate center f_0

$$\frac{a^2 e}{Eh^4}\int_0^a\int_0^b q(x, y)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\,dx\,dy =$$

$$= \frac{\pi^4}{48(1-\nu^2)}\left(e + \frac{1}{e}\right)^2 \zeta_0 + \frac{4234}{537{,}2\,(e + e^{-1})^2 + 322{,}5}\zeta_0^3; \tag{9.28}$$

where

$$e = \frac{b}{a}, \qquad \zeta_0 = \frac{f_0}{h}. \tag{9.29}$$

Thus, the problem in the classical form is solved.

Now by examining the principal equations of the improved theory, equations (9.14) and (9.15) and the boundary conditions (9.19), we see that they are completely identical in form to the corresponding equations and boundary conditions of the classical theory. Therefore, for the problem under consideration for the case of the improved theory

$$F = A\left[\sin\frac{\lambda x}{a} - \operatorname{sh}\frac{\lambda x}{a} - \alpha\left(\cos\frac{\lambda x}{a} - \operatorname{ch}\frac{\lambda x}{a}\right)\right]\times$$

$$\times\left[\sin\frac{\lambda y}{b} - \operatorname{sh}\frac{\lambda y}{b} - \alpha\left(\cos\frac{\lambda y}{b} - \operatorname{ch}\frac{\lambda y}{b}\right)\right], \qquad (9.30)$$

$$w = f\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \quad \zeta = \frac{f}{h}, \qquad (9.31)$$

$$A = -\frac{33f^2Eh}{537{,}2(e^{-2}+e^2)+322{,}5}, \qquad (9.32)$$

$$\frac{a^2e}{Eh^4}\int_0^a\int_0^b Q(x,y)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\,dx\,dy =$$

$$= \frac{\pi^4}{48(1-\nu^2)}(e+e^{-1})^2\zeta + \frac{4234}{537{,}2(e+e^{-1})^2+322{,}5}\zeta^3, \qquad (9.33)$$

where for Q(x, y) because of (9.26) and (9.16) we have

$$Q(x,y) =$$

$$= q(x,y) + \frac{DEh^3\pi^6}{10(1-\nu^2)G'b^6}\zeta_0(e^6+3e^4+3e^2+1)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \qquad (9.34)$$

ζ_0 is determined from (9.28).

Substituting values of Q(x, y) from (9.34) into (9.33) a final equation is obtained which establishes the relationship between the arbitrary transverse load q(x, y) and the deflection coefficient f. For the case when the plate is subjected to a uniformly distributed load of intensity q_0, i.e., when $q = q_0$ = constant, on the basis of (9.33) and (9.34) we have

$$q^* + \frac{\pi^8h^2}{1920(1-\nu^2)^2a^2}\cdot\frac{e^6+3e^4+3e^2+1}{m^6}\cdot\frac{E}{G'}\zeta_0 =$$

$$= \frac{\pi^6}{192(1-\nu^2)}(1+e^{-2})^2\zeta + \frac{10\,447}{537{,}2(1+e^4)+322{,}5e^2}\zeta^3, \qquad (9.35)$$

and to determine ζ_0 we have from (9.28)

$$q^* = \frac{\pi^6}{192(1-\nu^2)}(1+e^{-2})^2\zeta_0 + \frac{10\,447}{537{,}2(1+e^4)+322{,}5e^2}\zeta_0^3. \qquad (9.36)$$

In these equations along with the notations (9.29) and (9.31) the following is also used

$$q^* = \frac{q_0 a^4}{Eh^4}. \tag{9.37}$$

In the case of a square plate (e = 1) from (9.35) and (9.36) we have

$$q^* + \frac{\pi^8 h^2}{240(1-\nu^2)^2 a^2}\frac{E}{G'}\zeta_0 = \frac{\pi^6}{48(1-\nu^2)}\zeta + 7{,}48\zeta^3, \tag{9.38}$$

$$q^* = \frac{\pi^6}{48(1-\nu^2)}\zeta_0 + 7{,}48\zeta_0^3. \tag{9.39}$$

The table below shows values of the relative deflection ζ at different values of q* and relative thickness $h^* = Eh^2/G'(1-\nu^2)a^2$. All calculations were carried out with equations (9.38) and (9.39) and $\nu = 0.3$. The last column shows the values of ζ_0.

q^* \ h^*	0,50	0,20	0,10	0,05	0,00
20,0	1,15	0,92	0,83	0,79	0,76
40,0	1,62	1,38	1,30	1,25	1,22
60,0	1,90	1,69	1,61	1,57	1,53

It is evident from this table that the correction to the classical theory could be significant for certain types of anisotropy of the plate material.

With values of w(x, y), w_0(x, y) and F(x, y) it is easy to determine all other pertinent quantities for the problem using formulas (2.8.56) – (2.8.62).

10. Axisymmetric Problems of the Bending of Circular Transversely Isotropic Plates in the Presence of Large Displacements

Consider the axisymmetric bending of a circular plate of radius a, made of transversely isotropic material in such a way that the plane of anisotropy of the material is parallel to the middle plane at each point of the plate (Figure 16).

The vertical elements of the plate edge are rigidly fixed along the circumference at the mid-plane, but the plate edge can move freely within the mid-plane. Furthermore, the plate is loaded by a uniformly distributed load in such a way that

$$Z_1 = 0, \qquad Z_2 = q. \tag{10.1}$$

The problem is solved with equations (2.8.47) – (2.8.50), without considering the effect of normal stress σ_z ($A_1 = 0$, $A_2 = 0$), by the method of perturbations [46]. The method of perturbations has been previously used for the solution of the problem under consideration [47].

After the introduction of the dimensionless parameters

$$\left.\begin{aligned} \rho = \frac{r}{a}, \quad q^* = \frac{qa^4}{2Dh}, \quad w^* = \frac{w}{h}, \\ F^* = \frac{F}{Eh^3}, \quad h^* = \frac{Eh^2}{G'a^2}, \quad \sigma_i^* = \frac{\sigma_i a^2}{Eh^2} \end{aligned}\right\} \tag{10.2}$$

the system of equations (2.8.47) – (2.8.50) for the problem under consideration (see (1.4.10), (1.4.11), (2.5.1), (2.5.2) and (2.10)) takes the following form after some manipulations

$$\left.\begin{aligned} \rho\frac{d^3w^*}{d\rho^3} + \frac{d^2w^*}{d\rho^2} - \frac{1}{\rho}\frac{dw^*}{d\rho} = 12(1-\nu^2)\frac{dw^*}{d\rho}\frac{dF^*}{d\rho} + q^*\rho^2 - \\ - \frac{6}{5}h^*\rho\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\frac{dw^*}{d\rho}\frac{dF^*}{d\rho}\right)\right] \\ \rho\frac{d^3F^*}{d\rho^3} + \frac{d^2F^*}{d\rho^2} - \frac{1}{\rho}\frac{dF^*}{d\rho} = -\frac{1}{2}\left(\frac{dw^*}{d\rho}\right)^2. \end{aligned}\right\} \tag{10.3}$$

On the basis of equations (2.3.21) and (2.3.19) and assuming that $z_0 = 0$, the boundary conditions are:

at $\rho = 1$

$$w^* = 0, \quad -\frac{h}{a}\frac{dw^*}{dr} + a_r\frac{h^2}{8}\varphi = 0, \quad T_r = 0. \tag{10.4}$$

In addition we have the conditions for the plate center ($\rho = 0$) that $dw/dr = 0$ and T_r is a finite quantity.

During the derivation of the governing system of equations (10.3) the condition $\psi \equiv 0$ (see Section 6 of this Chapter) was used as well as equation (2.8.47) which for this problem is

$$\frac{d}{dr}(r\varphi) + \frac{12}{h^3}\frac{d}{dr}\left(\frac{dw}{dr}\frac{dF}{dr}\right) = -\frac{12}{h^3}qr. \tag{10.5}$$

Using equation (10.5) and considering the relationships (2.8.45) and (10.4), and the limiting condition on the function $\varphi(r)$ at the plate center ($r = 0$), the boundary conditions (10.4) are

at $\rho = 1$

$$w^* = 0, \quad \frac{dw^*}{d\rho} = -\frac{h^*q^*}{8(1-\nu^2)}, \quad \frac{1}{\rho}\frac{dF^*}{d\rho} = 0. \tag{10.6}$$

Thus, the problem is reduced to the solution of (10.3) for the boundary conditions (10.6). A dimensionless deflection of the plate center $\zeta = w^*(0)$ is used here as the principal perturbation parameter.

Functions $w^*(\rho)$ and $F^*(\rho)$ as well as the uniformly distributed load $q^* =$ constant are represented as infinite series in powers of ζ:

$$\left.\begin{aligned} w^*(\rho) &= w_1(\rho)\zeta + w_3(\rho)\zeta^3 + \ldots, \\ F^*(\rho) &= F_2(\rho)\zeta^2 + F_4(\rho)\zeta^4 + \ldots, \\ q^* &= \alpha_1\zeta + \alpha_3\zeta^3 + \alpha_5\zeta^5 + \ldots \end{aligned}\right\} \tag{10.7}$$

In the expressions for w^* and q^* terms are omitted with even powers of ζ and in the expression for F those with odd powers are omitted because they are otherwise eliminated during the derivation.

Substituting values of w^*, F^* and q^* from (10.7) into (10.3) and comparing terms with identical powers of ζ, a system of linear differential equations with respect to $w_i(\rho)$ and $F_j(\rho)$ ($i = 1, 3, \ldots, j = 2, 4, \ldots$) is obtained. In order to solve the system of differential equations thus obtained, the boundary conditions (10.6) must also be expanded into a power series in ζ and then the corresponding terms of this expansion are taken for each approximation.

Comparing the coefficients of ζ to the first power the equation for the first approximation is obtained:

$$\frac{d^3w_1}{d\rho^3} + \frac{1}{\rho}\frac{d^2w_1}{d\rho^2} - \frac{1}{\rho^2}\frac{dw_1}{d\rho} = \alpha_1\rho. \tag{10.8}$$

The boundary conditions and conditions at the plate center used with equation (10.8) are

$$w_1(1) = 0, \quad \left(\frac{dw_1}{d\rho}\right)_{\rho=1} = -\frac{h^*\alpha_1}{8\mu_0}, \quad \left(\frac{dw_1}{d\rho}\right)_{\rho=0} = 0, \tag{10.9}$$

as well as the equality

$$w_1(0) = 1. \tag{10.10}$$

Solving equation (10.8) for the conditions (10.9) and (10.10), we obtain the expansion coefficient $w_1(\rho)$ and the constant α_1

$$w_1 = \frac{\mu_0}{\mu_2}\rho^4 - \frac{2\mu_1}{\mu_2}\rho^2 + 1, \quad \alpha_1 = \frac{32\mu_0}{\mu_2}. \tag{10.11}$$

The following notations are used here:

$$\mu_0 = (1 - \nu^2), \quad \mu_1 = (1 - \nu^2 + h^*), \quad \mu_2 = (1 - \nu^2 + 2h^*). \tag{10.12}$$

It is easy to observe from (10.11) that we are dealing here with the small deflection problem. In particular, by assuming that $h^* = 0$ a solution is obtained which is identical with that for the clamped plate using the classical equations.

Equating now the coefficients of ζ^2, the equation for the second approximation is obtained

$$\frac{d^3F_2}{d\rho^3}+\frac{1}{\rho}\frac{d^2F_2}{d\rho^2}-\frac{1}{\rho^2}\frac{dF_2}{d\rho}=-\frac{8}{\mu_2^2}(\mu_0^2\rho^5-2\mu_0\mu_1\rho^3+\mu_1^2\rho). \quad (10.13)$$

The boundary conditions and the conditions at the plate center are

$$\left(\frac{1}{\rho}\frac{dF_2}{d\rho}\right)_{\rho=1}=0, \quad \left(\frac{1}{\rho}\frac{dF_2}{d\rho}\right)_{\rho=0}<\infty. \quad (10.14)$$

The solution of (10.13) with the conditions (10.14) is

$$\frac{dF_2}{d\rho}=-\frac{1}{\mu_2^2}\left[\frac{\mu_0^2}{6}\rho^7-\frac{2\mu_0\mu_1}{3}\rho^5+\mu_1^2\rho^3-\left(\frac{\mu_0^2}{6}-\frac{2}{3}\mu_0\mu_1+\mu_1^2\right)\rho\right]. \quad (10.15)$$

The equation for the third approximation is

$$\frac{d^3w_3}{d\rho^3}+\frac{1}{\rho}\frac{d^2w_3}{d\rho^2}-\frac{1}{\rho^2}\frac{dw_3}{d\rho}-12\mu_0\frac{1}{\rho}\frac{dw_1}{d\rho}\frac{dF_2}{d\rho}+$$
$$+\frac{6}{5}h^*\frac{d}{d\rho}\left[\frac{1}{\rho}\frac{d}{d\rho}\left(\frac{dw_1}{d\rho}\frac{dF_2}{d\rho}\right)\right]=\alpha_3\rho, \quad (10.16)$$

where

$$\frac{dw_1}{d\rho}\frac{dF_2}{d\rho}=-\frac{4\rho}{\mu_2^3}\left[\frac{\mu_0^3}{6}\rho^9-\frac{5\mu_0\mu_1}{6}\rho^7+\frac{5\mu_0\mu_1^2}{3}\rho^5-\right.$$
$$\left.-\left(\frac{\mu_0^3}{6}-\frac{2\mu_0^2\mu_1}{3}+\mu_0\mu_1^2+\mu_1^3\right)\rho^3+\left(\frac{\mu_0^2\mu_1}{6}-\frac{2\mu_0\mu_1^2}{3}+\mu_1^3\right)\rho\right]. \quad (10.17)$$

The boundary conditions and conditions at the plate center for w_3 are

$$w_3(1)=0, \quad \left(\frac{dw_3}{d\rho}\right)_{\rho=1}=-\frac{h^*\alpha_3}{8\mu_0}, \quad \left(\frac{dw_3}{d\rho}\right)_{\rho=0}=0. \quad (10.18)$$

Considering the condition (10.10) it must be assumed that $w_3(0)=0$.

Solving equation (10.16) with the conditions (10.18) we obtain

$$w_3=\frac{4{,}8h^*}{\mu_2^3}\left[\frac{\mu_0^3}{60}\rho^{10}-\frac{5\mu_0^2\mu_1}{48}\rho^8+\frac{5\mu_0\mu_1^2}{18}\rho^6-\right.$$
$$\left.-\frac{1}{4}\left(\frac{\mu_0^3}{6}-\frac{2\mu_0^2\mu_1}{3}+\mu_0\mu_1^2+\mu_1^3\right)\rho^4+\right.$$

$$+\frac{1}{2}\left(\frac{\mu_0^2\mu_1}{6}-\frac{2\mu_0\mu_1^2}{3}+\mu_1^3\right)\rho^2\Big]-\frac{48\mu_0}{\mu_1^3}\Big[\frac{\mu_0^3}{8640}\rho^{12}-$$

$$-\frac{\mu_0^2\mu_1}{960}\rho^{10}+\frac{5\mu_0\mu_1^2}{1152}\rho^8-\frac{1}{144}\left(\frac{\mu_0^3}{6}-\frac{2\mu_0^2\mu_1}{3}+\mu_0\mu_1^2+\mu_1^3\right)\rho^6+$$

$$+\frac{1}{32}\left(\frac{\mu_0^2\mu_1}{6}-\frac{2\mu_0\mu_1^2}{3}+\mu_1^3\right)\rho^4\Big]+\frac{\alpha_3}{32}\rho^4+\frac{\omega_1}{4}\rho^2, \qquad (10.19)$$

$$\alpha_3=\frac{38{,}4\mu_0 h^*}{\mu_2^2}\Big[-\frac{\mu_0^3}{10}+\frac{7\mu_0^2\mu_1}{12}-\frac{11\mu_0\mu_1^2}{9}+\mu_1^3\Big]-$$

$$-\frac{16\mu_0^2}{3\mu_2^4}\Big[\frac{\mu_0^3}{2}-\frac{89\mu_0^2\mu_1}{30}+\frac{25\mu_0\mu_1^2}{4}-5\mu_1^3\Big], \qquad (10.20)$$

$$\omega_1=-\frac{\mu_1}{4\mu_0}\alpha_3+\frac{8\mu_0}{\mu_2^3}\Big[-\frac{\mu_0^3}{15}+\frac{11\mu_0^2\mu_1}{24}-\frac{13\mu_0\mu_1^2}{12}+\mu_1\Big]. \qquad (10.21)$$

The fourth approximation is used to determine F_4. The equation for the fourth approximation is

$$\frac{d^3F_4}{d\rho^3}+\frac{1}{\rho}\frac{d^2F_4}{d\rho^2}-\frac{1}{\rho^2}\frac{dF_4}{d\rho}=-\frac{1}{\rho}\frac{dw_1}{d\rho}\frac{dw_3}{d\rho}. \qquad (10.22)$$

Substituting values of w_1 and w_3 from (10.11) and (10.19) and then solving equation (10.22) with the conditions

$$\left(\frac{1}{\rho}\frac{dF_4}{d\rho}\right)_{\rho=1}=0, \quad \left(\frac{1}{\rho}\frac{dF_4}{d\rho}\right)_{\rho=0}<\infty, \qquad (10.23)$$

a final expression for the expansion coefficient $F_4(\rho)$ is obtained, which is not given here because of its size.

If we use only these four approximations, then the following expression is obtained for the ratio between the dimensionless deflection ζ and the load:

$$\alpha_1\zeta+\alpha_3\zeta^3=q^*, \qquad (10.24)$$

where the coefficients α_i are given in (10.11) and (10.20).

For the radial stresses in the middle surface $\sigma_{r,u}^*$ we obtain, on the basis of (2.8.35) and (2.8.45):

at the plate center ($\rho = 0$)

$$\sigma_{r,c}^*(0,z)=\frac{1}{\mu_2^2}\left(\frac{\mu_0^2}{6}-\frac{2\mu_0\mu_1}{3}+\mu_1^2\right)\zeta^2+\frac{\omega_2}{2}\zeta^4, \qquad (10.25)$$

$$\overset{*}{\sigma}_{r,u}\left(0,-\frac{h}{2}\right)=-\frac{1+h^*\overset{*}{\sigma}_{r,c}(0,z)}{2(1-\nu)}\left\{\frac{4\mu_1}{\mu_2}\zeta-\right. \tag{10.26}$$

$$\left.-\left[\frac{4{,}8h^*}{\mu_2^3}\left(\frac{\mu_0^2\mu_1}{6}-\frac{2\mu_0\mu_1^2}{3}+\mu_1^3\right)+\frac{\omega_1}{2}\right]\zeta^3\right\}+\frac{h^*q^*}{24(1-\nu)(1-\nu^2)},$$

at the plate edge ($\rho = 1$) (conditionally)

$$\overset{*}{\sigma}_{r,c}(1,z)=0, \tag{10.27}$$

$$\overset{*}{\sigma}_{r,u}\left(1,-\frac{h}{2}\right)=\frac{2}{\mu_0\mu_2}(3\mu_0-\mu_1)\zeta+$$

$$+\left[\frac{2{,}4h^*}{\mu_0\mu_2^3}\left(\mu_0^3-\frac{11\mu_0^2\mu_1}{3}+\frac{14\mu_0\mu_1^2}{3}-2\mu_1^3\right)-\right.$$

$$\left.-\frac{4}{\mu_2^3}\left(-\frac{7\mu_0^3}{60}+\frac{31\mu_0^2\mu_1}{48}-\frac{31\mu_0\mu_1^2}{24}+\mu_1^3\right)+\frac{3\alpha_3}{16\mu_0}+\frac{\omega_1}{4\mu_0}\right]\zeta^3-$$

$$-\frac{h^*q^*}{16\mu_0^2}\left[\frac{2-\nu}{3}-h^*\left(\frac{d\overset{*}{\sigma}_{r,c}}{d\rho}\right)_{\rho=1}\right], \tag{10.28}$$

where along with (10.20) and (10.21) we have

$$\omega_2=\frac{12{,}8h^*}{\mu_1^4}\left[-\frac{5\mu_0^4}{672}+\frac{7\mu_0^3\mu_1}{120}-\frac{19\mu_0^2\mu_1^2}{96}+\frac{\mu_0\mu_1^3}{3}-\frac{\mu_1^4}{4}\right]-$$

$$-\frac{\mu_0}{\mu_2}\left[-\frac{13\mu_0^4}{20}+\frac{23\mu_0^3\mu_1}{70}-\frac{23\mu_0^2\mu_1^2}{18}+\frac{23\mu_0\mu_1^3}{10}-\frac{5\mu_1^4}{3}\right]+$$

$$+\frac{\alpha_3(\mu_0-2\mu_1)}{48\mu_2}+\frac{\omega_1(\mu_0-3\mu_1)}{6\mu_2}. \tag{10.29}$$

The tables below present computed values for different h* obtained from equations (10.24) – (10.28). The first rows in these tables (h* = 0) correspond to the problem as treated by the classical theory

ζ

h^* \ q^*	10,0	20,0	30,0
0,0	0,307	0,587	0,831
0,5	0,605	1,048	1,374

(10.30)

$\sigma^*_{r,\,c}(0, z)$

h^* \ q^*	10,0	20,0	30,0
0,0	0,047	0,169	0,330
0,5	0,120	0,318	0,466

(10.31)

$\sigma^*_{r.\,u}(0, -h/2)$

h^* \ q^*	10,0	20,0	30,0
0,0	—0,866	—1,599	—2,153
0,5	—0,803	—1,095	—0,643

(10.32)

We see from these tables that the dimensionless deflection and midplane stresses increase with h*, as compared with results obtained from the classical theory. In the case of bending stresses, the classical theory produces values that are too large.

These remarks are the result of the consideration of only the first four approximations of the method of perturbations and are valid only for investigations of the magnitude of changes of h* and q*.

11. Thermoelastic Problem for Orthotropic Plates

Consider an orthotropic plate in a rectangular Cartesian coordinate system x, y, z (see Section 5, Chapter II). We assume that the elastic properties of the material and the coefficients of thermal expansion $a_1 = a_x$ and $a_2 = a_y$ follow given functions of temperature over the plate, T(x, y, z).

The general theory of anisotropic plates as well as the generalized theory of F. Neiman [48, 49] for orthotropic plates are used here for the thermoelastic problem. This problem, with some generalization, was studied in certain other works [50, 51] and we will follow the course of their presentations.

Because of the absence of external surface loads ($X^{\pm} = Y^{\pm} = Z^{\pm} = 0$), on the basis of (2.2.24) and considering (2.4.1), (2.4.2), (2.2.27) and (2.2.29), it is easy to see that the normal stress σ_z is zero. (Here and in succeeding pages, the problem under consideration will be presented without any additional references; all necesary details can be found in the works of S. A. Ambartsumyan, S. M. Durgar'yan, and V. Novatskii.)

From the generalized Hooke's law we have for stresses

$$\left.\begin{aligned} \sigma_x = B_{11}e_x + B_{12}e_y + Q_{1t}T,\ \sigma_y = B_{22}e_y + B_{12}e_x + Q_{2t}T, \\ \tau_{xy} = B_{66}e_{xy},\ \tau_{xz} = B_{55}e_{xz},\ \tau_{yz} = B_{44}e_{yz}, \end{aligned}\right\} \quad (11.1)$$

where along with (2.5.2) we have

$$Q_{1t} = -(B_{11}\alpha_1 + B_{12}\alpha_2), \qquad Q_{2t} = -(B_{22}\alpha_2 + B_{12}\alpha_1); \quad (11.2)$$

At the same time we should remember that

$$\left.\begin{aligned} B_{ik} = B_{ik}(T),\ T = T(x, y, z),\ B_{ik} = B_{ik}(x, y, z), \\ \alpha_i = \alpha_i(T),\ \alpha_i = \alpha_i(x, y, z),\ Q_{iT} = Q_{iT}(x, y, z). \end{aligned}\right\} \quad (11.3)$$

On the basis of (1.2.2) because of the assumptions above, and considering (2.2.1), (2.2.2), (2.2.12), (1.4.5) and (2.4.1), we have inplane and normal displacements of any point of the plate:

$$\left.\begin{aligned} u_x = u - z\frac{\partial w}{\partial x} + \varphi\Gamma_1, \\ u_y = v - z\frac{\partial w}{\partial y} + \psi\Gamma_2, \end{aligned}\qquad u_z = w,\right\} \quad (11.4)$$

where

$$\left.\begin{aligned} \Gamma_1(x, y, z) = \frac{1}{2}\int_0^z a_{55}\left(\frac{h^2}{4} - z^2\right)dz, \\ \Gamma_2(x, y, z) = \frac{1}{2}\int_0^z a_{44}\left(\frac{h^2}{4} - z^2\right)dz. \end{aligned}\right\} \quad (11.5)$$

Considering (2.52) and (11.3)

$$a_{44} = a_{44}(x, y, z), \qquad a_{55} = a_{55}(x, y, z). \quad (11.6)$$

Substituting values of e_x, e_y, and e_{xy} from (1.2.2) into (11.1) and considering (11.4) we obtain for the stresses:

$$\sigma_x = B_{11}\frac{\partial u}{\partial x} + B_{12}\frac{\partial v}{\partial y} - z\left(B_{11}\frac{\partial^2 w}{\partial x^2} + B_{12}\frac{\partial^2 w}{\partial y^2}\right) + \\ + B_{11}\frac{\partial}{\partial y}(\varphi\Gamma_1) + B_{12}\frac{\partial}{\partial x}(\psi\Gamma_2) + Q_1T, \quad (11.7)$$

$$\sigma_y = B_{22}\frac{\partial v}{\partial y} + B_{12}\frac{\partial u}{\partial x} - z\left(B_{22}\frac{\partial^2 w}{\partial y^2} + B_{12}\frac{\partial^2 w}{\partial x^2}\right) + \\ + B_{22}\frac{\partial}{\partial y}(\psi\Gamma_2) + B_{12}\frac{\partial}{\partial x}(\varphi\Gamma_1) + Q_{2T}T, \quad (11.8)$$

$$\tau_{xy} = B_{66}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z\frac{\partial^2 w}{\partial x\,\partial y} + \frac{\partial}{\partial y}(\varphi\Gamma_1) + \frac{\partial}{\partial x}(\psi\Gamma_2)\right]. \quad (11.9)$$

With these expressions for stresses it is not difficult to express the internal stress and moment resultants in terms of the quantities u, v, φ, and ψ.

After certain substitutions and manipulations we obtain:

$$T_x = C_{11}\frac{\partial u}{\partial x} + C_{12}\frac{\partial v}{\partial y} - K_{11}\frac{\partial^2 w}{\partial x^2} - K_{12}\frac{\partial^2 w}{\partial y^2} + C_{11,x}\varphi + \\ + C_{122,y}\psi + C_{111}\frac{\partial \varphi}{\partial x} + C_{122}\frac{\partial \psi}{\partial y} + \theta_{10}, \quad (11.10)$$

$$T_y = C_{22}\frac{\partial v}{\partial y} + C_{12}\frac{\partial u}{\partial x} - K_{22}\frac{\partial^2 w}{\partial y^2} - K_{12}\frac{\partial^2 w}{\partial x^2} + C_{222,y}\psi + \\ + C_{121,x}\varphi + C_{222}\frac{\partial \psi}{\partial y} + C_{121}\frac{\partial \varphi}{\partial x} + \theta_{20}, \quad (11.11)$$

$$S = C_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - 2K_{66}\frac{\partial^2 w}{\partial x\,\partial y} + C_{661,y}\varphi + \\ + C_{662,x}\psi + C_{661}\frac{\partial \varphi}{\partial y} + C_{662}\frac{\partial \psi}{\partial x}, \quad (11.12)$$

$$N_x = \frac{h^3}{12}\varphi, \qquad N_y = \frac{h^3}{12}\psi, \quad (11.13)$$

$$M_x = -D_{11}\frac{\partial^2 w}{\partial x^2} - D_{12}\frac{\partial^2 w}{\partial y^2} + K_{11}\frac{\partial u}{\partial x} + K_{12}\frac{\partial v}{\partial y} + \\ + K_{111,x}\varphi + K_{122,y}\psi + K_{111}\frac{\partial \varphi}{\partial x} + K_{122}\frac{\partial \psi}{\partial y} + \theta_{11}, \quad (11.14)$$

$$M_y = -D_{22}\frac{\partial^2 w}{\partial y^2} - D_{12}\frac{\partial^2 w}{\partial x^2} + K_{22}\frac{\partial v}{\partial y} + K_{12}\frac{\partial u}{\partial x} + \\ + K_{222,y}\psi + K_{121,x}\varphi + K_{222}\frac{\partial \psi}{\partial y} + K_{121}\frac{\partial \varphi}{\partial x} + \theta_{21}, \quad (11.15)$$

$$H = -2D_{66}\frac{\partial^2 w}{\partial x\,\partial y} + K_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + K_{661,y}\varphi + \\ + K_{662,x}\psi + K_{661}\frac{\partial \varphi}{\partial y} + K_{662}\frac{\partial \psi}{\partial x}. \quad (11.16)$$

The different types of stiffnesses are denoted as follows:

$$\left.\begin{aligned} C_i &= \int_{-h/2}^{h/2} B_i dz, \quad K_i = \int_{-h/2}^{h/2} B_i z\, dz, \quad D_i = \int_{-h/2}^{h/2} B_i z^2\, dz, \\ C_{ij} &= \int_{-h/2}^{h/2} B_i \Gamma_j dz, \quad K_{ij} = \int_{-h/2}^{h/2} B_i \Gamma_j z\, dz, \\ C_{ij,r} &= \int_{-h/2}^{h/2} B_i \frac{\partial \Gamma_j}{\partial r} dz, \quad K_{ij,r} = \int_{-h/2}^{h/2} B_i \frac{\partial \Gamma_j}{\partial r} z\, dz \\ &(i = 11, 12, 22, 66; \quad j = 1, 2; \quad r = x, y), \end{aligned}\right\} \tag{11.17}$$

as well as

$$\theta_{ij} = \int_{-h/2}^{h/2} Q_{iT} T z^j dz \qquad (i = 1, 2; \quad j = 0, 1). \tag{11.18}$$

Substituting the expressions for the internal stress and moment resultants (11.10) – (11.16) into the equilibrium equations of the plate element (2.2.45) – (2.2.49), the following system of five differential equations with variable coefficients with respect to the five undetermined functions is obtained:

$$\begin{aligned} &\left[\frac{\partial}{\partial x}\left(C_{11}\frac{\partial}{\partial x}\right)+\frac{\partial}{\partial y}\left(C_{66}\frac{\partial}{\partial y}\right)\right]u+\left[\frac{\partial}{\partial x}\left(C_{12}\frac{\partial}{\partial y}\right)+\frac{\partial}{\partial y}\left(C_{66}\frac{\partial}{\partial x}\right)\right]v- \\ &-\left[\frac{\partial}{\partial x}\left(K_{11}\frac{\partial^2}{\partial x^2}+K_{12}\frac{\partial^2}{\partial y^2}\right)+2\frac{\partial}{\partial y}\left(K_{66}\frac{\partial^2}{\partial x\,\partial y}\right)\right]w+ \\ &+\left[\frac{\partial}{\partial x}\left(C_{111}\frac{\partial}{\partial x}+C_{111,x}\right)+\frac{\partial}{\partial y}\left(C_{661}\frac{\partial}{\partial y}+C_{661,y}\right)\right]\varphi+ \\ &+\left[\frac{\partial}{\partial x}\left(C_{122}\frac{\partial}{\partial y}+C_{122,y}\right)+\frac{\partial}{\partial y}\left(C_{662}\frac{\partial}{\partial x}+C_{662,x}\right)\right]\psi=-\frac{\partial\theta_{10}}{\partial x}, \end{aligned} \tag{11.19}$$

$$\begin{aligned} &\left[\frac{\partial}{\partial x}\left(C_{66}\frac{\partial}{\partial y}\right)+\frac{\partial}{\partial y}\left(C_{12}\frac{\partial}{\partial x}\right)\right]u+\left[\frac{\partial}{\partial x}\left(C_{66}\frac{\partial}{\partial x}\right)+\frac{\partial}{\partial y}\left(C_{22}\frac{\partial}{\partial y}\right)\right]v- \\ &-\left[\frac{\partial}{\partial y}\left(K_{22}\frac{\partial^2}{\partial y^2}+K_{12}\frac{\partial^2}{\partial x^2}\right)+2\frac{\partial}{\partial x}\left(K_{66}\frac{\partial^2}{\partial x\,\partial y}\right)\right]w+ \end{aligned}$$

$$+\left[\frac{\partial}{\partial y}\left(C_{222}\frac{\partial}{\partial y}+C_{222,\,y}\right)+\frac{\partial}{\partial x}\left(C_{662}\frac{\partial}{\partial x}+C_{662,\,x}\right)\right]\psi+$$

$$+\left[\frac{\partial}{\partial y}\left(C_{121}\frac{\partial}{\partial x}+C_{121,\,x}\right)+\frac{\partial}{\partial x}\left(C_{661}\frac{\partial}{\partial y}+C_{661,\,y}\right)\right]\varphi=-\frac{\partial\theta_{20}}{\partial y},\tag{11.20}$$

$$\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y}=0,\tag{11.21}$$

$$\left[\frac{\partial}{\partial x}\left(K_{11}\frac{\partial}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{66}\frac{\partial}{\partial y}\right)\right]u+\left[\frac{\partial}{\partial x}\left(K_{12}\frac{\partial}{\partial y}\right)+\frac{\partial}{\partial y}\left(K_{66}\frac{\partial}{\partial x}\right)\right]v-$$

$$-\left[\frac{\partial}{\partial x}\left(D_{11}\frac{\partial^2}{\partial x^2}+D_{12}\frac{\partial^2}{\partial y^2}\right)+2\frac{\partial}{\partial y}\left(D_{66}\frac{\partial^2}{\partial x\,\partial y}\right)\right]w+$$

$$+\left[\frac{\partial}{\partial x}\left(K_{111}\frac{\partial}{\partial x}+K_{111,\,x}\right)+\frac{\partial}{\partial y}\left(K_{661}\frac{\partial}{\partial y}+K_{661,\,y}\right)-\frac{h^3}{12}\right]\varphi+$$

$$+\left[\frac{\partial}{\partial x}\left(K_{122}\frac{\partial}{\partial y}+K_{122,\,y}\right)+\frac{\partial}{\partial y}\left(K_{662}\frac{\partial}{\partial x}+K_{662,\,x}\right)\right]\psi=-\frac{\partial\theta_{11}}{\partial x},\tag{11.22}$$

$$\left[\frac{\partial}{\partial y}\left(K_{22}\frac{\partial}{\partial y}\right)+\frac{\partial}{\partial x}\left(K_{66}\frac{\partial}{\partial x}\right)\right]v+\left[\frac{\partial}{\partial y}\left(K_{12}\frac{\partial}{\partial x}\right)+\frac{\partial}{\partial x}\left(K_{66}\frac{\partial}{\partial y}\right)\right]u-$$

$$-\left[\frac{\partial}{\partial y}\left(D_{22}\frac{\partial^2}{\partial y^2}+D_{12}\frac{\partial^2}{\partial x^2}\right)+2\frac{\partial}{\partial x}\left(D_{66}\frac{\partial^2}{\partial x\,\partial y}\right)\right]w+$$

$$+\left[\frac{\partial}{\partial y}\left(K_{222}\frac{\partial}{\partial y}+K_{222,\,y}\right)+\frac{\partial}{\partial x}\left(K_{662}\frac{\partial}{\partial x}+K_{662,\,x}\right)-\frac{h^3}{12}\right]\psi+$$

$$+\left[\frac{\partial}{\partial y}\left(K_{121}\frac{\partial}{\partial x}+K_{121,\,x}\right)+\frac{\partial}{\partial x}\left(K_{661}\frac{\partial}{\partial y}+K_{661,\,y}\right)\right]\varphi=-\frac{\partial\theta_{21}}{\partial y}.\tag{11.23}$$

In addition to equations (11.19) – (11.23), we have boundary conditions which are defined in the usual way (see (2.3.13) - (2.3.18)).

Cylindrical bending. Consider the problem of cylindrical bending due to temperature. It is assumed that the boundary conditions and the temperature changes are such that the plate bends into a cylindrical surface.

The y-axis is directed along the long side of the plate (Figure 12) and the boundary conditions along the plate edges x = 0, x = ℓ remain unchanged. The temperature is a function of only the z and x coordinates, T = T(x, z). It can then be assumed that $v \equiv 0$, $\psi \equiv 0$ and the remaining quantities of the problem, u, w, φ are functions of only the independent variable x (see Sections 1 and 8 of this Chapter). Due to this, the system of differential equations (11.19) – (11.23) for the cylindrical bending of the plate is

$$\frac{d}{dx}\left(C_{11}\frac{du}{dx}\right)-\frac{d}{dx}\left(K_{11}\frac{d^2w}{dx^2}\right)+\frac{d}{dx}\left(C_{111}\frac{d}{dx}+C_{111,x}\right)\varphi=-\frac{d\theta_{10}}{dx}, \tag{11.24}$$

$$\frac{d\varphi}{dx}=0, \quad \varphi=A_1, \tag{11.25}$$

$$\frac{d}{dx}\left(K_{11}\frac{du}{dx}\right)-\frac{d}{dx}\left(D_{11}\frac{d^2w}{dx^2}\right)+ \\ +\left[\frac{d}{dx}\left(K_{111}\frac{d}{dx}+K_{111,x}\right)-\frac{h^3}{12}\right]\varphi=-\frac{d\theta_{11}}{dx}. \tag{11.26}$$

Considering identities (11.25), the following expressions are obtained from (11.24) and (11.26) after some elementary transformations (here $C_{11}D_{11}-K_{11}^2\neq 0$):

$$\frac{du}{dx}=\frac{1}{C_{11}D_{11}-K_{11}^2}\left[A_1\left(K_{11}K_{111,x}-D_{11}C_{111,x}-K_{11}\frac{h^3}{12}x\right)+\right. \\ \left.+A_2D_{11}-A_3K_{11}-D_{11}\theta_{10}+K_{11}\theta_{11}\right], \tag{11.27}$$

$$\frac{d^2w}{dx^2}=\frac{1}{C_{11}D_{11}-K_{11}^2}\left[A_1\left(C_{11}K_{111;x}-K_{11}C_{111,x}-C_{11}\frac{h^3}{12}x\right)+\right. \\ \left.+A_2K_{11}-A_3C_{11}-K_{11}\theta_{10}+C_{11}\theta_{11}\right], \tag{11.28}$$

where A_1, A_2, A_3 are integration constants.

It is not difficult to solve equations (11.27) and (11.28) provided the temperature changes and the functions which represent the dependence of the elastic and thermoelastic characteristics on temperature are known.

Six integration constants (three from (11.27) and (11.28)) and another three appearing in the course of solution, are present in the expressions for φ, u, and w and are determined from the boundary conditions along the edges x = 0 and x = l.

We consider a numerical example [52]. Let the temperature at the lower plate plane (z = h/2) be changed accordingly in a linear manner from 0° to 400° C, and at the upper plane (z = –h/2) let the temperature be maintained at 0° (Figure 19), i.e., we have

$$\left.\begin{array}{l}\text{at } z = h/2 \\ \qquad T_{\text{н}} = 400\dfrac{x}{l}, \\ \text{at } z = -h/2 \\ \qquad T_{\text{в}} = 0.\end{array}\right\} \tag{11.29}$$

Assuming that

$$T = 400\frac{x}{l}\left(\frac{z}{h}+\frac{1}{2}\right), \tag{11.30}$$

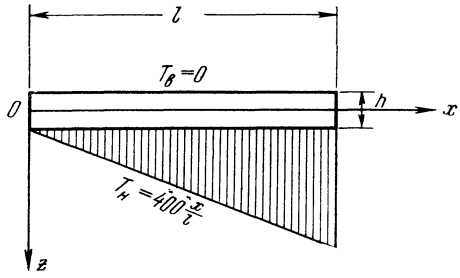

Figure 19.

it is easy to see that the temperature function T satisfies both the equation of thermal conductivity and the conditions (11.29). It is further assumed that

$$\left.\begin{aligned} &E_x = E(1-0{,}0005T),\quad G_{xz} = G'(1-0{,}0005T),\\ &\alpha_1 = \alpha_2 = \alpha = \text{const},\quad \nu_1 = \text{const},\ \nu_2 = \text{const}. \end{aligned}\right\} \tag{11.31}$$

Furthermore, the plate edges are fixed in such a way that

$$\left.\begin{aligned} &\text{at } x = 0,\ z = 0\\ &\qquad u_x = u_z = \frac{\partial u_z}{\partial x} = 0,\\ &\text{at } x = l,\ z = 0\\ &\qquad u_x = u_z = \frac{\partial u_z}{\partial x} = 0. \end{aligned}\right\} \tag{11.32}$$

Expressions (11.30) – (11.32) are sufficient for the solution of the numerical example under consideration. Using the quantities (11.30) – (11.32), the basic formulas (11.2) – (11.5), (11.17) and (11.18) and also equations (11.25), (11.27) and (11.28) we obtain for the integration constants (after some cumbersome manipulation)

$$A_1 = -\frac{337{,}4}{1-0{,}001163h^*}\frac{E^*}{hl}, \qquad A_4 = A_5 = A_6 = 0,$$

$$A_2 = -\frac{92{,}59+0{,}1481h^*}{1-0{,}001163h^*}E^*h, \qquad A_3 = -\frac{0{,}6365+0{,}2798h^*}{1-0{,}001163h^*}E^*h^2,$$

where

$$h^* = \frac{E}{G'(1-\nu_1\nu_2)}\frac{h^2}{l^2}, \qquad E^* = \frac{1+\nu_2}{1-\nu_1\nu_2}E\alpha.$$

With the integration constants A_i available, the problem is solved.

From the values of A_i we see that the correction due to transverse shear can reach a significant size. For example, for the case $h^* = 0.114$ to 1.00 the correction due in the determination of A_2 may reach 5 to 40%.

12. Bending of a Simply-Supported Two-Layer Plate by a Sinusoidal Load

Consider a rectangular (a x b) simply-supported two-layer ($\delta_1 = \delta_2 = \delta$) plate made of transversely isotropic layers (Figures 10, 13).

Let the plate be loaded by a normally applied load distributed according to

$$Z_1 = 0, \qquad Z_2 = Z = q\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \tag{12.1}$$

where q is the intensity of the load at the plate center (x = a/2, y = b/2).

This problem may be solved on the basis of the theory presented in Section 9, Chapter II.

Let the plate layers have the following properties:

(a) for the first layer (where a prime sign is used instead of m)

$$\left.\begin{aligned} B_{11}' = B_{22}' &= \frac{nE}{1-\nu^2}, \qquad B_{12}' = \nu\frac{nE}{1-\nu^2}, \\ B_{66}' &= \frac{nE}{2(1+\nu)}, \qquad B_{55}' = B_{44}' = \frac{1}{k}B_{11}'; \end{aligned}\right\} \tag{12.2}$$

(b) for the second layer (where two prime signs are used instead of m)

$$\left.\begin{aligned} B_{11}'' = B_{22}'' &= \frac{E}{1-\nu^2}, \; B_{12}'' = \nu\frac{E}{1-\nu^2}, \\ B_{66}'' &= \frac{E}{2(1+\nu)}, \qquad B_{55}'' = B_{44}'' = \frac{1}{k}B_{11}''. \end{aligned}\right\} \tag{12.3}$$

In these expressions the proportionality constants k and n are introduced along with the usual notations.

Substituting the values of B_{ik} from (12.2) and (12.3) into (2.9.11) – (2.9.13) and considering that $\delta_1 = \delta_2 = \delta$ we obtain for the stiffnesses:

$$\left.\begin{aligned} C = C_{11} = C_{22} &= \frac{E\delta(1+n)}{1-\nu^2}, \quad C_{12} = \nu C, \quad C_{66} = \frac{1-\nu}{2}C, \\ K = K_{11} = K_{22} &= \frac{E\delta^2(1-n)}{2(1-\nu^2)}, \quad K_{12} = \nu K, \quad K_{66} = \frac{1-\nu}{2}K, \\ D = D_{11} = D_{22} &= \frac{E\delta^3(1+n)}{3(1-\nu^2)}, \quad D_{12} = \nu D, \quad D_{66} = \frac{1-\nu}{2}D. \end{aligned}\right\} \quad (12.4)$$

Next, for the parameters χ_i, ζ_i φ_i we obtain from (2.9.16) – (2.9.18) on the basis of (2.9.5) – (2.9.10), (12.2), and (12.3):

$$\left.\begin{aligned} \chi_1'^{,0} = \chi_1''^{,0} &= -k\left(\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial x\,\partial y^2}\right)w_0, \\ \chi_2'^{,0} = \chi_2''^{,0} &= -k\left(\frac{\partial^3}{\partial y^3} + \frac{\partial^3}{\partial x^2\,\partial y}\right)w_0; \end{aligned}\right\} \quad (12.5)$$

$$\left.\begin{aligned} \zeta_1'^{,0} = \zeta_1''^{,0} &= k\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u_0}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 v_0}{\partial x\,\partial y}\right), \\ \zeta_2'^{,0} = \zeta_2''^{,0} &= k\left(\frac{\partial^2 v_0}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v_0}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 u_0}{\partial x\,\partial y}\right); \end{aligned}\right\} \quad (12.6)$$

$$\left.\begin{aligned} \varphi_1'^{,0} &= -\delta\zeta_1'^{,0} + \frac{\delta^2}{2}\chi_1'^{,0}, \quad \varphi_1''^{,0} = \delta\zeta_1'^{,0} + \frac{\delta^2}{2}\chi_1'^{,0}, \\ \varphi_2'^{,0} &= -\delta\zeta_2'^{,0} + \frac{\delta^2}{2}\chi_2'^{,0}, \quad \varphi_2''^{,0} = \delta\zeta_2'^{,0} + \frac{\delta^2}{2}\chi_2'^{,0}. \end{aligned}\right\} \quad (12.7)$$

Finally, because of (12.5) – (12.7) we have from (2.9.39) and (2.9.40) for the parameters R_i^m and P_i^m

$$\left.\begin{aligned} R_1' &= -\frac{\delta^3}{3}\zeta_1'^{,0} + \frac{5\delta^4}{24}\chi_1'^{,0}, \quad R_1'' = \frac{\delta^3}{3}\zeta_1'^{,0} + \frac{5\delta^4}{24}\chi_1'^{,0}, \\ R_2' &= -\frac{\delta^3}{3}\zeta_2'^{,0} + \frac{5\delta^4}{24}\chi_2'^{,0}, \quad R_2'' = \frac{\delta^3}{3}\zeta_2'^{,0} + \frac{5\delta^4}{24}\chi_2'^{,0}, \end{aligned}\right\} \quad (12.8)$$

$$\left.\begin{aligned} P_1' &= -\frac{5\delta^4}{24}\zeta_1'^{,0} + \frac{2\delta^5}{15}\chi_1'^{,0}, \quad P_1'' = \frac{5\delta^4}{24}\zeta_1'^{,0} + \frac{2\delta^5}{15}\chi_1'^{,0}, \\ P_2' &= -\frac{5\delta^4}{24}\zeta_2'^{,0} + \frac{2\delta^5}{15}\chi_2'^{,0}, \quad P_2 = \frac{5\delta^4}{24}\zeta_2'^{,0} + \frac{2\delta^5}{15}\chi_2'^{,0}. \end{aligned}\right\} \quad (12.9)$$

From equations (12.5) – (12.9) we see that all the necessary parameters are functions of the displacements of the classical theory, i.e., $u_0 = u_0(x, y)$, $v_0 = v_0(x, y)$, $w_0 = w_0(x, y)$, which may be determined from the system of equations

(2.9.4). Considering (2.9.5) – (2.9.10) the equations (2.9.4) may be written:

$$\left.\begin{aligned}
&\frac{\partial^2 u_0}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2 u_0}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2 v_0}{\partial x\,\partial y}-\frac{K}{C}\left(\frac{\partial^3 w_0}{\partial x^3}+\frac{\partial^3 w_0}{\partial x\,\partial y^2}\right)=0,\\
&\frac{\partial^2 v_0}{\partial y^2}+\frac{1-\nu}{2}\frac{\partial^2 v_0}{\partial x^2}+\frac{1+\nu}{2}\frac{\partial^2 u_0}{\partial x\,\partial y}-\frac{K}{C}\left(\frac{\partial^3 w_0}{\partial y^3}+\frac{\partial^3 w_0}{\partial x^2\,\partial y}\right)=0,\\
&\frac{\partial^4 w_0}{\partial x^4}+2\frac{\partial^4 w_0}{\partial x^2\,\partial y^2}+\frac{\partial^4 w_0}{\partial y^4}-\\
&\qquad-\frac{K}{D}\left(\frac{\partial^3 u_0}{\partial x^3}+\frac{\partial^3 u_0}{\partial x\,\partial y^2}+\frac{\partial^3 v_0}{\partial y^3}+\frac{\partial^3 v_0}{\partial y\,\partial x^2}\right)=\frac{Z}{D}.
\end{aligned}\right\}\quad(12.10)$$

From the system of equations (12.10), by eliminating displacements u_0 and v_0, we have

$$\Delta\Delta w_0=\frac{Z}{D^*}. \qquad (12.11)$$

Here

$$D^*=D-\frac{K^2}{C}=\frac{E\delta^3}{12(1-\nu^2)}\,\frac{1+14n+n^2}{1+n}, \qquad (12.12)$$

$$\Delta(\quad)=\frac{\partial^2(\quad)}{\partial x^2}+\frac{\partial^2(\quad)}{\partial y^2}. \qquad (12.13)$$

Next, using the first two equations of the system (12.10), the parameters ζ R_i, P_i from (12.6), (12.8) and (12.9) are:

$$\zeta_1'^{,0}=\zeta_1''^{,0}=k\frac{K}{C}\frac{\partial}{\partial x}\Delta w_0, \qquad \zeta_2'^{,0}=\zeta_2''^{,0}=k\frac{K}{C}\frac{\partial}{\partial y}\Delta w_0; \qquad (12.14)$$

$$\left.\begin{aligned}
&R_1'=-A_1\frac{\partial}{\partial x}\Delta w_0, \qquad R_1''=-A_2\frac{\partial}{\partial x}\Delta w_0,\\
&R_2'=-A_1\frac{\partial}{\partial y}\Delta w_0, \qquad R_2''=-A_2\frac{\partial}{\partial y}\Delta w_0,\\
&A_1=k\frac{\delta^4}{24}\frac{9+n}{1+n}, \qquad A_2=k\frac{\delta^4}{24}\frac{1+9n}{1+n};
\end{aligned}\right\}\quad(12.15)$$

$$\left.\begin{aligned} P_1' &= -B_1 \frac{\partial}{\partial x}\Delta w_0, \quad P_1'' = -B_2 \frac{\partial}{\partial x}\Delta w_0, \\ P_2' &= -B_1 \frac{\partial}{\partial y}\Delta w_0, \quad P_2'' = -B_2 \frac{\partial}{\partial y}\Delta w_0, \\ B_1 &= k\frac{\delta^5}{240}\frac{57+7n}{1+n}, \quad B_2 = k\frac{\delta^5}{240}\frac{7+57n}{1+n} \end{aligned}\right\} \tag{12.16}$$

Substituting the values of R_i and P_i from (12.15) and (12.16) into (2.9.37) and (2.9.38) we obtain the correction terms:

$$\left.\begin{aligned} T_x^* &= -T\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)\Delta w_0, \\ T_y^* &= -T\left(\frac{\partial^2}{\partial y^2} + \nu\frac{\partial^2}{\partial x^2}\right)\Delta w_0, \\ S &= -T(1-\nu)\frac{\partial^2}{\partial x\,\partial y}\Delta w_0, \end{aligned}\right\} \tag{12.17}$$

where

$$T = \frac{E}{1-\nu^2}k(1-n)\frac{\delta^4}{24}, \tag{12.18}$$

$$\left.\begin{aligned} M_x^* &= -M\left(\frac{\partial^2}{\partial x^2} + \nu\frac{\partial^2}{\partial y^2}\right)\Delta w_0, \\ M_y^* &= -M\left(\frac{\partial^2}{\partial y^2} + \nu\frac{\partial^2}{\partial x^2}\right)\Delta w_0, \\ H^* &= -M(1-\nu)\frac{\partial^2}{\partial x\,\partial y}\Delta w_0, \end{aligned}\right\} \tag{12.19}$$

and where

$$M = \frac{E}{1-\nu^2}k\frac{7+114n+7n^2}{1+n}\frac{\delta^5}{240}. \tag{12.20}$$

From equations (2.9.5) – (2.9.10), (12.17) – (12.19) and (2.9.42) – (2.9.44) a system of three differential equations with respect to the three displacements can now be obtained:

$$\left.\begin{aligned} C\left(\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial x\,\partial y}\right) - K\frac{\partial}{\partial x}\Delta w &= T\frac{\partial}{\partial x}\Delta\Delta w_0, \\ C\left(\frac{\partial^2 v}{\partial y^2} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial x\,\partial y}\right) - K\frac{\partial}{\partial y}\Delta w &= T\frac{\partial}{\partial y}\Delta\Delta w_0, \\ D\Delta\Delta w - K\left(\frac{\partial}{\partial x}\Delta u + \frac{\partial}{\partial y}\Delta v\right) &= Z - M\Delta\Delta\Delta w_0. \end{aligned}\right\} \tag{12.21}$$

From the system (12.21), by eliminating displacements u and v, the following fourth order differential equation with respect to the function w is obtained:

$$\Delta\Delta w = \frac{Z}{D^*} - k\frac{1+62n+n^2}{1+14n+n^2}\frac{\delta^2}{10}\Delta\Delta\Delta w_0. \tag{12.22}$$

The value of w_0, which is present in (12.22) is determined from (12.11).

The solution of equations (12.11) and (12.22) is sought in the form:

$$w_0 = f_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \tag{12.23}$$

$$w = f \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \tag{12.24}$$

The conditions of simple supports (2.44) are, evidently, completely satisfied by (12.23) and (12.24).

Substituting values of w_0, w and Z from (12.23), (12.24) and (12.1), respectively, into (12.11) and (12.22) we have for the coefficients f_0 and f:

$$f_0 = \frac{q}{D^*}\frac{1}{\left(\frac{\pi^2}{a^2}+\frac{\pi^2}{b^2}\right)^2}, \tag{12.25}$$

$$f = f_0\left[1 + k\frac{1+62n+n^2}{1+14n+n^2}\frac{\delta^2}{10}\left(\frac{\pi^2}{a^2}+\frac{\pi^2}{b^2}\right)\right]. \tag{12.26}$$

When values of f_0 and f from (12.25) and (12.26) are substituted into (12.23) and (12.24), the solutions for w_0 and w are obtained. Thus, f_0 is the parameter which characterizes the classical solution and f characterizes the solution obtained by the theory suggested here. The second term in parenthesis in (12.26) is, in fact, a correction to the classical theory which results from consideration of the phenomena related to transverse shear.

All remaining quantities of interest can be easily found since the expression for w is known.

The results obtained here agree well with those obtained from a more refined theory [53].

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CHAPTER IV

SOME PROBLEMS OF VIBRATION AND STABILITY OF PLATES

1. Free Vibrations

This section discusses briefly some problems of transverse vibration of orthotropic and transversely isotropic plates.

Consider a plate in the orthogonal system of coordinates (x, y, z) with axes coincident with the principal directions of elasticity of the plate material. The density of the plate material is ρ.

The plate is loaded by a normally applied load

$$\left.\begin{aligned} Z_1 = \frac{1}{2}(Z^+ - Z^-), \quad Z_2 = Z^+ + Z^-, \\ X^+ = X^- = 0, \quad Y^+ = Y^- = 0, \quad X_i = Y_i = 0, \end{aligned}\right\} \tag{1.1}$$

and for f(z) we take (2.4.1), i.e.,

$$f_i(z) = f(z) = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right). \tag{1.2}$$

The equations of motion (1.2.3) can be presented in the following form when there are no body forces:

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = \rho \frac{\partial^2 u_x}{\partial t^2}, \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = \rho \frac{\partial^2 u_y}{\partial t^2}, \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \rho \frac{\partial^2 u_z}{\partial t^2}; \end{aligned}\right\} \tag{1.3}$$

where t is the time.

For this problem from identities (2.2.2), (2.2.12), (2.2.18), (2.2.19) because of (1.2) and (2.4.2) we have

$$\tau_{xz} = f(z)\varphi(x,y,t), \qquad \tau_{yz} = f(z)\psi(x,y,t), \tag{1.4}$$

and

$$\left.\begin{aligned} u_z &= w(x, y, t) \\ u_x &= u(x, y, t) - z\frac{\partial w}{\partial x} + a_{55}\frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\varphi, \\ u_y &= v(x, y, t) - z\frac{\partial w}{\partial y} + a_{44}\frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\psi. \end{aligned}\right\} \tag{1.5}$$

Solving the third equation of the system (1.3) and considering (1.1) – (1.5) and (2.2.25) we obtained for σ_z:

$$\sigma_z = Z_1 - \frac{z}{2}\left(\frac{h^2}{4} - \frac{z^2}{3}\right)\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\right) + z\rho\frac{\partial^2 w}{\partial t^2}. \tag{1.6}$$

Regarding other stresses, because of (2.2.3) – (2.2.5), (1.2.2), (1.5), (1.6), as well as (2.5.1) and (2.5.2), we have

$$\sigma_x = \bar{\sigma}_x - zA_1\rho\frac{\partial^2 w}{\partial t^2}, \quad \sigma_y = \bar{\sigma}_y - zA_2\rho\frac{\partial^2 w}{\partial t^2}, \quad \tau_{xy} = \bar{\tau}_{xy},$$

where the quantities $\bar{\sigma}_x, \bar{\sigma}_y, \bar{\tau}_{xy}$ represent corresponding stresses which are determined without consideration of the inertial terms, i.e., (2.5.15), (2.5.16) and (2.5.17).

After multiplication of all three equations of (1.3) by dz and also the first two by z dz, and after integration from –h/2 to h/2, five equations of motion are obtained in terms of internal stress and moment resultants.

$$\left.\begin{aligned} &\frac{\partial T_x}{\partial x} + \frac{\partial S}{\partial y} = \rho h\frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial T_y}{\partial y} + \frac{\partial S}{\partial x} = \rho h\frac{\partial^2 v}{\partial t^2}, \\ &\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} = -Z_2 + \rho h\frac{\partial^2 w}{\partial t^2}, \\ &\frac{\partial M_x}{\partial x} + \frac{\partial H}{\partial y} - N_x = -\rho\frac{h^3}{12}\frac{\partial^3 w}{\partial x\,\partial t^2} + \rho a_{55}\frac{h^5}{120}\frac{\partial^2 \varphi}{\partial t^2}, \\ &\frac{\partial M_y}{\partial y} + \frac{\partial H}{\partial x} - N_y = -\rho\frac{h^3}{12}\frac{\partial^3 w}{\partial y\,\partial t^2} + \rho a_{44}\frac{h^5}{120}\frac{\partial^2 \psi}{\partial t^2}. \end{aligned}\right\} \tag{1.7}$$

Internal stress resultants T_x (x, y, t), T_y(x, y, t), S(x, y, t), N_x(x, y, t), N_y(x, y, t) and torque resultant H(x, y, t) are determined from formulas (2.5.8) – (2.5.10), (2.5.14) and (2.5.13); for bending moment resultants M_x(x, y, t), M_y(x, y, t) we have

$$M_x = \bar{M}_x - \rho A_1\frac{h^3}{12}\frac{\partial^2 w}{\partial t^2}, \quad M_y = \bar{M}_y - \rho A_2\frac{h^3}{12}\frac{\partial^2 w}{\partial t^2}, \tag{1.8}$$

where M_x and M_y are determined from (2.5.11) and (2.5.12).

Substituting values of internal stress and moment resultants into (1.7), equations of motion in terms of the displacements u(x, y, t), v(x, y, t), w(x, y, t) and the shear functions $\varphi(x, y, t)$ and $\psi(x, y, t)$ are obtained:

$$C_{11}\frac{\partial^2 u}{\partial x^2}+C_{66}\frac{\partial^2 u}{\partial y^2}+(C_{12}+C_{66})\frac{\partial^2 v}{\partial x\,\partial y}=A_1 h\frac{\partial Z_1}{\partial x}+\rho h\frac{\partial^2 u}{\partial t^2}, \quad (1.9)$$

$$C_{22}\frac{\partial^2 v}{\partial y^2}+C_{66}\frac{\partial^2 v}{\partial x^2}+(C_{12}+C_{66})\frac{\partial^2 u}{\partial x\,\partial y}=A_2 h\frac{\partial Z_1}{\partial y}+\rho h\frac{\partial^2 v}{\partial t^2}, \quad (1.10)$$

$$\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y}=-\frac{12}{h^3}Z_2+\rho h\frac{12}{h^3}\frac{\partial^2 w}{\partial t^2}, \quad (1.11)$$

$$D_{11}\frac{\partial^3 w}{\partial x^3}+(D_{12}+2D_{66})\frac{\partial^3 w}{\partial x\,\partial y^2}-\frac{h^2}{10}\left[a_{55}\left(D_{11}\frac{\partial^2}{\partial x^2}+D_{66}\frac{\partial^2}{\partial y^2}\right)\varphi+\right.$$
$$\left.+a_{44}(D_{12}+D_{66})\frac{\partial^2\psi}{\partial x\,\partial y}\right]+\frac{h^3}{12}\varphi-A_1\frac{h^5}{120}\frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y}\right)=$$
$$=(1-A_1)\rho\frac{h^3}{12}\frac{\partial^3 w}{\partial x\,\partial t^2}-\rho a_{55}\frac{h^5}{120}\frac{\partial^2\varphi}{\partial t^2}, \quad (1.12)$$

$$D_{22}\frac{\partial^3 w}{\partial y^3}+(D_{12}+2D_{66})\frac{\partial^3 w}{\partial y\,\partial x^2}-\frac{h^2}{10}\left[a_{44}\left(D_{22}\frac{\partial^2}{\partial y^2}+D_{66}\frac{\partial^2}{\partial x^2}\right)\psi+\right.$$
$$\left.+a_{55}(D_{12}+D_{66})\frac{\partial^2\varphi}{\partial x\,\partial y}\right]+\frac{h^3}{12}\psi-A_2\frac{h^5}{120}\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}+\frac{\partial\varphi}{\partial x}\right)=$$
$$=(1-A_2)\rho\frac{h^3}{12}\frac{\partial^3 w}{\partial y\,\partial t^2}-\rho a_{44}\frac{h^5}{120}\frac{\partial^2\psi}{\partial t^2}, \quad (1.13)$$

where

$$C_{ik}=hB_{ik}, \qquad D_{ik}=\frac{h^3}{12}B_{ik}. \quad (1.14)$$

Thus we have obtained the equations of motion in terms of the functions u, v, w, φ, ψ. Boundary conditions (2.3.13) – (2.3.18) should be added to these equations, as well as the initial conditions

at t = 0

$$\left.\begin{aligned} w&=w_0(x,y), & \frac{\partial w}{\partial t}&=w_1(x,y),\\ u&=u_0(x,y), & \frac{\partial u}{\partial t}&=u_1(x,y),\\ v&=v_0(x,y), & \frac{\partial v}{\partial t}&=v_1(x,y), \end{aligned}\right\} \quad (1.15)$$

where u_0, v_0, w_0 ; u_1, v_1, w_1, are, respectively, known components of the initial displacements and the initial velocity for the point (x, y).

It is easy to detect wave characteristics in the system of equations (1.9) – (1.13) after performing a series of transformations. However, this is not shown here because the wave characteristics of the plate are beyond our interest in the text to follow. We will be dealing only with the determination of the frequencies of low-frequency vibrations of the plate which can be found with sufficient accuracy without a consideration of the rotational inertia [1]. Therefore, all terms which owe their origin to rotational inertia are omitted from the system of equations (1.9) – (1.13) when we deal with actual problems.

As has been indicated by a comparative analysis [2] the improved theories will produce a satisfactory approximation to the more precise theory for the first three series of frequencies. The best approximation is obtained from the theory presented in this book and from the theory of E. Reissner [3].

Some cases of free transverse vibrations of plates are discussed below.

1. Determination of vibration frequency of simply-supported (along the entire contour) rectangular orthotropic plates. The initial system of differential equations of plate vibrations (1.9) – (1.13) is considerably simplified when all terms which account for rotational inertia and normal stress σ_z are omitted and only transverse vibrations without inplane displacements are considered.

   After an appropriate elementary transformation we obtain from (1.9) – (1.13) [4]

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} = \frac{12}{h^3} \frac{\gamma}{g} h \frac{\partial^2 w}{\partial t^2},$$

$$\left[B_{11} \frac{\partial^3}{\partial x^3} + (B_{12} + 2B_{66}) \frac{\partial^3}{\partial x \partial y^2} \right] w - \frac{h^2}{10} \left[a_{55} \left(B_{11} \frac{\partial^2}{\partial x^2} + \right. \right.$$

$$\left. \left. + B_{66} \frac{\partial^2}{\partial y^2} \right) \varphi + a_{44} (B_{12} + B_{66}) \frac{\partial^2 \psi}{\partial x \partial y} \right] + \varphi = 0,$$

$$\left[B_{22} \frac{\partial^3}{\partial y^3} + (B_{12} + 2B_{66}) \frac{\partial^3}{\partial y \partial x^2} \right] w - \frac{h^2}{10} \left[a_{44} \left(B_{22} \frac{\partial^2}{\partial y^2} + B_{66} \frac{\partial^2}{\partial x^2} \right) \psi + \right.$$

$$\left. + a_{55} (B_{12} + B_{66}) \frac{\partial^2 \varphi}{\partial y \partial x} \right] + \psi = 0, \qquad (1.16)$$

   where γ is the specific weight of the plate material; g represents the acceleration of gravity ($\rho = \gamma/g$).

   Boundary conditions of simple supports (2.3.14′) should be added to these equations, which may be presented in the form (Figure 13)

$$\begin{array}{lll} \text{at} & x = 0, & x = a \\ & & w = 0, \quad M_x = 0, \quad \psi = 0; \\ \text{at} & y = 0, & y = b \\ & & w = 0, \quad M_y = 0, \quad \varphi = 0. \end{array}$$

The solution of the system (1.16) is sought in the following form:

$$\left.\begin{aligned} w(x,y,t) &= f\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\omega_{mn}t,\\ \varphi(x,y,t) &= c_\varphi\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\omega_{mn}t,\\ \psi(x,y,t) &= c_\psi\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\cos\omega_{mn}t, \end{aligned}\right\} \tag{1.17}$$

which satisfies all the conditions of the simple supports along the plate contour. f, c_φ, c_ψ are constants; m and n are whole numbers.

Substituting the expressions for w, φ, ψ from (1.17) into (1.16), the following system of algebraic equations with respect to the unknown constants f, c_φ, c_ψ is obtained (after some simplifications):

$$\left.\begin{aligned} &\frac{\gamma}{g}\omega_{mn}^2 f-\frac{h^2}{12}\frac{m\pi}{a}c_\varphi-\frac{h^2}{12}\frac{n\pi}{b}c_\psi=0,\\ &\frac{m\pi^3}{a}\left[B_{11}\frac{m^2}{a^2}+(B_{12}+2B_{66})\frac{n^2}{b^2}\right]f-\left[a_{55}\frac{\pi^2h^2}{10}\left(B_{11}\frac{m^2}{a^2}+B_{66}\frac{n^2}{b^2}\right)+\right.\\ &\qquad\left.+1\right]c_\varphi-a_{44}\frac{\pi^2h^2}{10}(B_{12}+B_{66})\frac{mn}{ab}c_\psi=0,\\ &\frac{n\pi^3}{b}\left[B_{22}\frac{n^2}{b^2}+(B_{12}+2B_{66})\frac{m^2}{a^2}\right]f-a_{55}\frac{\pi^2h^2}{10}(B_{12}+B_{66})\frac{mn}{ab}c_\varphi-\\ &\qquad-\left[a_{44}\frac{\pi^2h^2}{10}\left(B_{22}\frac{n^2}{b^2}+B_{66}\frac{m^2}{a^2}\right)+1\right]c_\psi=0. \end{aligned}\right\} \tag{1.18}$$

In order to have nontrivial solutions of the system (1.18) it is necessary that the determinant be equal to zero. If the determinant of the system (1.18) is set equal to zero then we obtain the frequencies of natural vibration of the plate under consideration, expressed in the following form:

$$\omega_{mn}=\overset{0}{\omega}_{mn}\sqrt{1-d}, \tag{1.19}$$

where $\overset{0}{\omega}_{mn}$ is the frequency of natural vibration of a simply-supported (along the entire contour) orthotropic rectangular plate as found from the classical theory [5]. This classical solution can be expressed in the form:

$$\overset{0}{\omega}_{mn}=\pi^2\sqrt{\frac{g}{\gamma h}}\left[D_{11}\frac{m^4}{a^4}+D_{22}\frac{n^4}{b^4}+2(D_{12}+2D_{66})\frac{m^2n^2}{a^2b^2}\right]^{1/2}; \tag{1.20}$$

The term d, which characterizes the correction to the classical theory is

$$\left.\begin{aligned} d &= \frac{B-A}{1+B}, \\ A &= \frac{\pi^2 h^2}{10}\left(a_{44}\frac{m^2}{a^2}+a_{55}\frac{n^2}{b^2}\right)\times \\ &\times \frac{\left(B_{11}\frac{m^2}{a^2}+B_{66}\frac{n^2}{b^2}\right)\left(B_{66}\frac{m^2}{a^2}+B_{22}\frac{n^2}{b^2}\right)-(B_{12}+B_{66})^2\frac{m^2n^2}{a^2b^2}}{B_{11}\frac{m^4}{a^4}+2(B_{12}+2B_{66})\frac{m^2n^2}{a^2b^2}+B_{22}\frac{n^4}{b^4}}, \\ B &= \frac{\pi^2 h^2}{10}\left[a_{55}\left(B_{11}\frac{m^2}{a^2}+B_{66}\frac{n^2}{b^2}\right)+a_{44}\left(B_{66}\frac{m^2}{a^2}+B_{22}\frac{n^2}{b^2}\right)\right]+ \\ &+ a_{44}a_{55}\frac{\pi^4 h^4}{100}\left[\left(B_{11}\frac{m^2}{a^2}+B_{66}\frac{n^2}{b^2}\right)\left(B_{66}\frac{m^2}{a^2}+B_{22}\frac{n^2}{b^2}\right)-\right. \\ &\left.-(B_{12}+B_{66})^2\frac{m^2n^2}{a^2b^2}\right]. \end{aligned}\right\} \quad (1.21)$$

Thus, ω^0_{mn} in (1.19) represents the frequency of natural vibration of the plate as determined with the classical theory, while the second term is the correction to it due to the consideration of transverse shear.

From (1.21) we see that this correction is strongly dependent on not only the size and elastic characteristics of the plate but also on the number of half-waves in the x and y directions.

Considering the properties of the coefficients B_{ik} and a_{jj} we see from (1.21) that d is a positive value and is always less than unity. Consequently, the frequency of interest to us is determined from formula (1.19) and is always lower than the corresponding frequency found with the classical theory.

In the particular case when the plate is made of a transversely isotropic material (the plane of isotropy is parallel to the plate middle plane at each point) on the basis of (1.4.10), (1.4.11) and (2.5.2)

$$\left.\begin{aligned} &B_{11}=B_{22}=B_{12}+2B_{66}=\frac{E}{1-\nu^2}, \quad B_{12}=\frac{\nu E}{1-\nu^2}, \\ &B_{66}=G=\frac{E}{2(1+\nu)}, \quad B_{12}+B_{66}=\frac{E}{2(1-\nu)}, \\ &a_{55}=a_{44}=\frac{1}{G'}, \quad a_{13}=a_{23}=-\frac{\nu_{13}}{E_3}=-\frac{\nu_{23}}{E_3}=-\frac{\nu'}{E'}, \\ &A_1=A_2=a_{13}(B_{11}+B_{12})=a_{23}(B_{22}+B_{12})=-\frac{\nu' E}{E'(1-\nu)}, \\ &\nu_1=\nu_2=\nu, \quad \nu_{13}=\nu_{23}=\nu'. \end{aligned}\right\} \quad (1.22)$$

Then for vibration frequencies, from (1.19) and (1.20), we obtain

$$\omega_{mn} = \overset{0}{\omega}_{mn} \frac{1}{\sqrt{1 + h^*\left(\frac{m^2}{c^2} + n^2\right)}}, \tag{1.23}$$

$$\overset{0}{\omega}_{mn} = \frac{\pi^2}{b^2}\sqrt{\frac{gD}{\gamma h}}\left(\frac{m^2}{c^2} + n^2\right), \tag{1.24}$$

where the following notations are used

$$\left.\begin{aligned} h^* &= \frac{\pi^2 h^2}{10b^2}\frac{E}{G'(1-\nu^2)}, \\ D &= \frac{Eh^3}{12(1-\nu^2)} \\ c &= \frac{a}{b}. \end{aligned}\right\} \tag{1.25}$$

It should be remembered that frequencies (1.24), or (1.23) at h* = 0, are the result of the classical theory of plates and they have the same value for isotropic and transversely isotropic plates. This shows the indifference of the classical theory to changes of the elastic characteristics of the plate material in directions perpendicular to the plate middle plane.

The Table below shows the frequency change of the plate vibrations vs the parameter h*. Values of the ratio $\omega_{mn} / \overset{0}{\omega}_{mn}$, which corresponds to some vibration mode (m, n = 1, 2), have been calculated from (1.23) for square (c = 1) plates at different values of the parameter h*.

h^*	$\omega_{11}/\omega_{11}^0$	$\omega_{12}/\omega_{12}^0$	$\omega_{22}/\omega_{22}^0$
0,00	1,00	1,00	1,00
0,02	0,9806	0,9535	0,9285
0,03	0,9713	0,9325	0,8980
0,05	0,9535	0,8944	0,8452
0,07	0,9366	0,8607	0,8006
0,10	0,9129	0,8165	0,7454
0,15	0,8771	0,7559	0,6742
0,20	0,8452	0,7071	0,6202
0,25	0,8165	0,6667	0,5774
0,30	0,7906	0,6325	0,5423

It is evident from this Table that the frequency of natural vibrations, ω_{mn}, differs more and more from the corresponding frequencies, $\overset{0}{\omega}_{mn}$,

found from the classical theory, as the parameter h* increases. This difference is less prominent for the lowest frequency ω_{11}, but becomes more noticeable for the frequencies of higher modes, $\omega_{12} = \omega_{21}$, ω_{22}, etc.

2. Determination of the vibration frequency of transversely isotropic circular plates. Equations for the free transverse vibrations of transversely isotropic circular plates can be obtained if Z in (3.7.1) and (3.7.2) is replaced with

$$-\frac{\gamma h}{g}\frac{\partial^2 w}{\partial t^2}, \tag{1.26}$$

where γ is the specific weight of the plate material; g is the acceleration of gravity.

After this substitution the following differential equations are obtained:

$$\Delta\Delta w - \frac{\gamma h}{gD}(1 - k\Delta)\frac{\partial^2 w}{\partial t^2} = 0, \quad \Delta\Phi - \delta^2\Phi = 0. \tag{1.27}$$

where for D, Δ, k and δ^2 we have equations (3.7.3), (3.7.4), etc., respectively.

$$\left.\begin{aligned} \Delta &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \vartheta}\left(\frac{1}{r}\frac{\partial}{\partial \vartheta}\right)\right], \quad D = \frac{Eh^3}{12(1-\nu^2)}, \\ k &= \left(2\frac{G}{G'} - \nu'\frac{E}{E'}\right)\frac{h^2}{10(1-\nu^2)}, \quad \delta^2 = \frac{10}{h^2}\frac{G'}{G}. \end{aligned}\right\} \tag{1.28}$$

The solution of the system of equations (1.27) corresponding to the plate vibration with n nodal parameters is [6]

$$\left.\begin{aligned} w(r, \vartheta, t) &= W(r)\cos n\vartheta\cos\omega t, \\ \Phi(r, \vartheta, t) &= F(r)\sin n\vartheta\cos\omega t, \end{aligned}\right\} \tag{1.29}$$

where ω is the circular frequency of natural vibration; n = 0, 1, 2, 3, . . . is the number of waves of the plate middle surface in the circumferential direction.

Using the necessary information from the work cited above, we substitute w and Φ from (1.29) into (1.27). This leads to the following ordinary differential equations for the determination of W(r) and F(r):

$$\Delta_n\Delta_n W - \frac{\gamma h}{gD}\omega^2(1 - k\Delta_n)W = 0, \quad \Delta_n F - \delta^2 F = 0; \tag{1.30}$$

Here

$$\Delta_n = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}. \tag{1.31}$$

A general solution of equations (1.30) is

$$\left.\begin{aligned} W(r) &= c_1 J_n(\alpha r) + c_2 Y_n(\alpha r) + c_3 I_n(\beta r) + c_4 K_n(\beta r), \\ F(r) &= c_5 I_n(\delta r) + c_6 K_n(\delta r), \end{aligned}\right\} \tag{1.32}$$

where J_n, Y_n and I_n, K_n are Bessel functions with real and purely imaginary arguments:

$$\left.\begin{aligned} \alpha &= \left[\sqrt{\left(\frac{\gamma h \omega^2 k}{2gD}\right)^2 + \frac{\gamma h \omega^2}{gD}} + \frac{\gamma h k \omega^2}{2gD}\right]^{1/2}, \\ \beta &= \left[\sqrt{\left(\frac{\gamma h \omega^2 k}{2gD}\right)^2 + \frac{\gamma h \omega^2}{gD}} - \frac{\gamma h k \omega^2}{2gD}\right]^{1/2}, \end{aligned}\right\} \tag{1.33}$$

and where $c_1, \ldots, c_6$ are integration constants.

Let the plate of radius a be of a full circular shape (Figure 16). Because of this it should be assumed that $c_2 = c_4 = c_6 = 0$; then on the basis of (1.29) and (1.32)

$$\left.\begin{aligned} w(r, \vartheta, t) &= [c_1 J_n(\alpha r) + c_3 I_n(\beta r)] \cos n\vartheta \cos \omega t, \\ \Phi(r, \vartheta, t) &= c_5 I_n(\delta r) \sin n\vartheta \cos \omega t. \end{aligned}\right\} \tag{1.34}$$

The integration constants presented in (1.34) can be determined from conditions at the contour, r = a.

We consider two cases for the plate contour.

Case I. The plate is simply-supported along the contour. The boundary conditions of the simple supports are

at r = a

$$w = 0, \qquad M_r = 0, \qquad \psi = 0. \tag{1.35}$$

From equations (3.7.7), (3.7.8), (1.34) and the boundary conditions (1.35) a uniform system of algebraic equations in terms of the integration constants c_1, c_3, c_5 is obtained. When the determinant of this system is set equal to zero, the following transcendental equation is obtained for the determination of the frequencies of natural vibration of the plate:

$$(1-\nu)\left[\beta_0\frac{I_{n-1}(\beta_0)}{I_n(\beta_0)}-\alpha_0\frac{J_{n-1}(\alpha_0)}{J_n(\alpha_0)}\right]+$$
$$+\frac{2\alpha_0\beta_0}{\delta_0^2}\left[\alpha_0\frac{I_{n-1}(\beta_0)}{I_n(\beta_0)}+\beta_0\frac{J_{n-1}(\alpha_0)}{J_n(\alpha_0)}\right]-$$
$$-(\alpha_0^2+\beta_0^2)\left[1+\frac{2n}{\delta_0^2}\frac{I_{n-1}(\delta_0)}{I'_n(\delta_0)}\right]=0, \qquad (1.36)$$

where

$$\left.\begin{aligned} I'_n(x)&=\frac{dI_n}{dx}=I_{n-1}(x)-\frac{n}{x}I_n(x),\\ \alpha_0&=\alpha a, \quad \beta_0=\beta a, \quad \delta_0=\delta a,\end{aligned}\right\} \qquad (1.37)$$

The following relation between α and β exists because of (1.33):

$$\alpha^2\beta^2=\frac{\gamma n}{gD}\omega^2, \qquad \beta=\frac{\alpha}{\sqrt{1+k\alpha^2}}. \qquad (1.38)$$

The frequencies of natural vibration ω_{nj} for each value of n and j(n = 0, 1, 2, . . . ; j = 1, 2, . . .), using relations (1.38), are determined as the roots of equation (1.36) by the following formula:

$$\omega_{nj}=\sqrt{\frac{gD}{\gamma h}}\frac{\alpha_{nj}^2}{\sqrt{1+k\alpha_{nj}^2}}, \qquad (1.39)$$

where $a_{nj} = a_{onj}/a_j$; a_{onj} is the j-th root of equation (1.36) for a fixed value of n.

A corresponding transcendental equation of the classical theory can be obtained from (1.36). By assuming that k = 0 in (1.36) and passing to the limit $\delta_0 \to \infty$ we obtain

$$\frac{I_{n-1}(\alpha_0)}{I_n(\alpha_0)}-\frac{J_{n-1}(\alpha_0)}{J_n(\alpha_0)}=\frac{2\alpha_0}{1-\nu}. \qquad (1.36')$$

The corresponding frequencies of natural vibration can be determined from (1.39) with k = 0.

The roots of equation (1.36) and the corresponding vibration frequency for some values of the ratio E/G′ and h/a have been calculated from the above formulas and equations on the assumption that E/E′ = 0. The results obtained are presented in the next two Tables. These Tables show the value of the first two roots a_{onj} (j = 1, 2) in (1.36) and

corresponding to them values

$$\bar{\omega}_{nj} = \sqrt{\frac{\gamma h}{gD}}\,\omega_{nj} \tag{1.40}$$

for each of the values n = 0, 1, 2. In all calculations it was assumed that ν = 0.3.

The results in these Tables show that the frequencies of natural vibrations as calculated while including the effects of transverse shear differ noticeably from corresponding frequencies found with the classical theory (E/G′ = 0). This difference increases with an increase in the ratio E/G′ and h/a, and it increases for higher n and j.

$h/a = 1/10$

n	E/G′					
	0,0		2,6		5,0	
	α_{0nj}	$\bar{\omega}_{nj}$	α_{0nj}	$\bar{\omega}_{nj}$	α_{0nj}	$\bar{\omega}_{nj}$
0,0	2,222 5,452	4,935 29,720	2,223 5,454	4,906 28,559	2.224 5,453	4,879 27,599
1,0	3,728 6,963	13,898 48,479	3,730 6,966	13,643 45,473	3,731 6,968	13 419 43,143
2,0	5,061 8,374	25,613 70,117	5,063 8,377	24,746 64,051	5,065 8,380	24,020 59,654

$h/a = 1/5$

n	E/G′					
	0,0		2,6		5,0	
	α_{0nj}	$\bar{\omega}_{nj}$	α_{0nj}	$\bar{\omega}_{nj}$	α_{0nj}	$\bar{\omega}_{nj}$
0,0	2,222 5,452	4,935 29,720	2,226 5,461	4,821 25,752	2,230 5,467	4,722 23,219
1,0	3,728 6,963	13,898 48,479	3,735 6,973	12,953 38,986	3,740 6,980	12,233 33,855
2,0	5,061 8,374	25,613 70,117	5,069 8,385	22,594 52,356	5,076 8,391	20,585 44,117

Case II. The plate is rigidly clamped along the contour. The boundary conditions in this case on the basis of (2.3.15) or (2.3.21) are

at r = a

$$\left.\begin{aligned} w &= 0, \\ -\frac{\partial w}{\partial r} + \frac{1}{2G'}\left(\frac{h^2}{4} - \frac{z_0^2}{3}\right)\varphi &= 0, \\ -\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{1}{2G'}\left(\frac{h^2}{4} - \frac{z_0^2}{3}\right)\psi &= 0. \end{aligned}\right\} \tag{1.41}$$

Here, just as in the general theory, it is assumed that the conditions for which the tangential displacements u_r and u_θ are equal to zero are satisfied only along two circumferences $z = \pm z_0$ ($0 \leqslant z_0 \leqslant h/2$) of the lateral plate surface. Following then in the same way as in the case of a simply-supported plate, a transcendental equation is obtained for the determination of the frequencies of natural vibration of a clamped plate:

$$\begin{aligned} &\beta_0 \frac{I_{n-1}(\beta_0)}{I_n(\beta_0)} - \alpha_0 \frac{J_{n-1}(\alpha_0)}{J_n(\alpha_0)} + \\ &+ \frac{10}{\delta_0^2(1-\nu)}\left(\frac{1}{4} - \frac{z_0^2}{3h^2}\right)\left\{\alpha_0\beta_0\left[\alpha_0 \frac{I_{n-1}(\beta_0)}{I_n(\beta_0)} + \beta_0 \frac{J_{n-1}(\alpha_0)}{J_n(\alpha_0)}\right] - \right. \\ &\left. - n(\alpha_0^2 + \beta_0^2)\frac{I_{n-1}(\delta_0)}{I_n'(\delta_0)}\right\} = 0. \end{aligned} \tag{1.42}$$

Frequencies of natural vibrations, ω_{nj}, just as in the previous problem, are determined from (1.39) with the use of roots of the transcendental equation (1.42). Assuming that k = 0, and letting $\delta_0 \to \infty$, it is easy to obtain the corresponding equation of the classical theory:

$$J_n(\alpha_0) I_{n-1}(\alpha_0) - J_{n-1}(\alpha_0) I_n(\alpha_0) = 0. \tag{1.42'}$$

Here, as before, the roots of equation (1.42) have been computed and also the corresponding frequencies, for the same numerical values of E/G', at h/a = 1/5 and for two values of the ratio z_0/h. The results obtained are presented in the two Tables below.

$h/a = 1/5$, $z_0/h = 1/2$

E/G' / n	0,0		2,6		5,0	
	α_{0nj}	$\bar{\omega}_{nj}$	α_{0nj}	$\bar{\omega}_{nj}$	α_{0nj}	$\bar{\omega}_{nj}$
0,0	3,196 6,306	10,216 39,771	3,150 6,127	9,405 31,403	3,114 6,023	8,804 27,059
1,0	4,611 7,800	21,260 60,829	4,505 7,544	18,284 44,304	4,432 7,423	16,416 37,056
2,0	5,906 9,197	34,877 84,583	5,733 8,873	28,027 57,124	5,634 8,748	24,364 46,732

$h/a = 1/5$, $z_0/h = 1/10$

E/G' / n	0,0		2,6		5,0	
	α_{0nj}	$\bar{\omega}_{nj}$	α_{0nj}	$\bar{\omega}_{nj}$	α_{0nj}	$\bar{\omega}_{nj}$
0,0	3,196 6,306	10,216 39,771	3,116 6,013	9,213 30,415	3,055 5,862	8,501 25,935
1,0	4,611 7,800	21,260 60,829	4,436 7,397	17,778 42,914	4,323 7,237	15,733 35,708
2,0	5,906 9,197	34,877 84,583	5,632 8,704	27,171 55,463	5,490 8,556	23,376 45,320

It was assumed here that $\nu = 0.3$, $E/E' = 0$.

The results verify the conclusions of the previous problem. The frequencies of natural vibration of the plate increase with an increase in the ratio z_0/h in the problem under study, provided all other conditions are equal. This means that with an increase in distance between the clamped

circumferences of the lateral plate surface the general rigidity of the plate increases as well as the plate vibration frequency. However, the difference in frequencies at $z_0/h = 1/10$ and $z_0/h = 1/2$ is insignificant. This once more verifies the statement made in Section 1 of Chapter III for the case of cylindrical bending of a plate with a clamped edge.

2. Static Stability of Plates

In this section the determination of critical load is illustrated on certain actual examples without presenting all the details of the generally known statements and definitions on which the stability of elastic plates is based [7–9]. In addition, some formulas are presented which show the effect of transverse shear and the normal stress σ_z on values of the critical loads for the stability problems of orthotropic and transversely isotropic plates.

Let the plate be deformed by forces which act in its middle plane and which are small enough that the flat shape of the plate is the only possible configuration in equilibrium, and this equilibrium is stable. If these forces are increased, the initial flat form of the plate may become unstable and the plate may pass, under the effect of negligibly small perturbations, to a new stable equilibrium with a curved middle surface.

The transition of the plate from the initial form of equilibrium into a new curved form of equilibrium is called the loss of stability. The smallest loads which cause this transition are called the critical loads.

This description of the occurrence of instability is not complete and is not rigorous. More rigorous definitions can be found in other works [10] where the stability of elastic systems are presented on the basis of sufficiently general concepts.

By assuming that the initial state of stress in the plate involves no moments, we arrive at a fictitious transverse load of the form

$$-(T_x^0 \varkappa_1 + T_y^0 \varkappa_2 + S^0 \tau), \tag{2.1}$$

where T_x^0, T_y^0, S^0 are internal inplane stress resultants of the initial flat state of the plate; κ_1, κ_2, τ are the curvatures and the twist of the plate middle surface in the curved form of equilibrium for which we have also (see (2.2.21) – (2.2.23)):

$$\varkappa_1 = -\frac{\partial^2 w}{\partial x^2}, \quad \varkappa_2 = -\frac{\partial^2 w}{\partial y^2}, \quad \tau = -2\frac{\partial^2 w}{\partial x\, \partial y}. \tag{2.2}$$

Substituting values of κ_i, τ from (2.2) into (2.1) we obtain an expression for a fictitious normally applied load, which appears during the derivation of the force equilibrium equations. This expression is obtained for a small deviation from the flat equilibrium shape [11, 12].

$$\tilde{Z} = T_x^0 \frac{\partial^2 w}{\partial x^2} + T_y^0 \frac{\partial^2 w}{\partial y^2} + 2S^0 \frac{\partial^2 w}{\partial x\, \partial y}. \tag{2.3}$$

All other (tangential) components of the fictitious load are neglected in the problems discussed below.

1. Stability of simply-supported (along the entire contour) orthotropic rectangular plates.

   Consider the stability problem of a simply-supported (along the entire contour x = 0, x = a and y = 0, y = b) rectangular (a x b) orthotropic plate which is compressed within its own plane along two principal directions x and y. These directions are parallel to the plate edges (Figure 20).

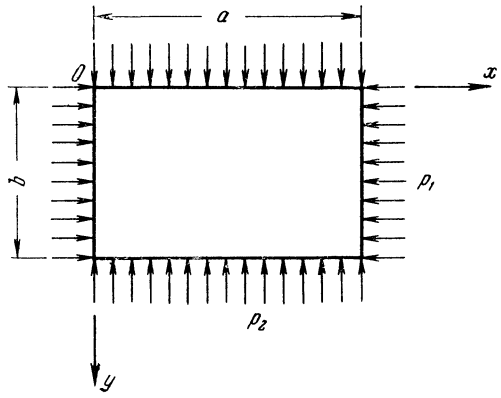

Figure 20.

Let the compressive forces have a uniform distribution with intensity p_1 in the x direction, and p_2 in the y direction. It is also assumed that their ratio is constant, i.e., $p_1 = p$, $p_2 = \lambda p$ (λ is some constant).

A critical value of the load needs to be determined, $p_{cr} = p^*$, at which the plate becomes unstable, i.e., at which the flat form of equilibrium ceases to be stable [13].

The differential equations for plate stability are obtained by generalizing the initial system of differential equations for transverse (u = 0, v = 0) plate bending, including the consideration of the effect of transverse shear (2.5.25) – (2.5.27) on the plate stability. For this purpose the intensity of the normally applied load Z_2 is understood to be the given fictitious load which appears because of the equilibrium conditions of the adjacent buckled form of equilibrium, i.e., $\widetilde{Z}$.

As was mentioned before, the plate is in a momentless state prior to the loss of stability. Therefore, using the initial plane stress state for the internal stress resultants T_x^0, T_y^0, S^0 we have

$$T_x^0 = -p, \qquad T_y^0 = -\lambda p, \qquad S^0 = 0. \tag{2.4}$$

On the basis of (2.3), (2.4) and (1.14) we obtain from (2.5.25) – (2.5.27) the following system of differential equations for the stability of an orthotropic plate including the effect of transverse shear:

$$\left.\begin{aligned}
&\frac{h^3}{12}\left(\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y}\right)-p\left(\frac{\partial^2 w}{\partial x^2}+\lambda\frac{\partial^2 w}{\partial y^2}\right)=0,\\
&\left[B_{11}\frac{\partial^3}{\partial x^3}+(B_{12}+2B_{66})\frac{\partial^3}{\partial x\,\partial y^2}\right]w-\\
&-\frac{h^2}{10}\left[a_{55}\left(B_{11}\frac{\partial^2}{\partial x^2}+B_{66}\frac{\partial^2}{\partial y^2}\right)\varphi+a_{44}(B_{12}+B_{66})\frac{\partial^2\psi}{\partial x\,\partial y}\right]+\varphi=0,\\
&\left[B_{22}\frac{\partial^3}{\partial y^3}+(B_{12}+2B_{66})\frac{\partial^3}{\partial y\,\partial x^2}\right]w-\\
&-\frac{h^2}{10}\left[a_{44}\left(B_{22}\frac{\partial^2}{\partial y^2}+B_{66}\frac{\partial^2}{\partial x^2}\right)\psi+a_{55}(B_{12}+B_{66})\frac{\partial^2\varphi}{\partial x\,\partial y}\right]+\psi=0.
\end{aligned}\right\}\tag{2.5}$$

The boundary conditions for simple supports should be added to equations (2.5), which may be presented as follows:

at x = 0	x = a
	$w = 0, M_x = 0, \psi = 0;$
at y = 0	y = b
	$w = 0, M_y = 0, \varphi = 0.$

The solution of the system (2.5) is:

$$\left.\begin{aligned}
w(x,y) &= f\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},\\
\varphi(x,y) &= c_\varphi\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b},\\
\psi(x,y) &= c_\psi\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b},
\end{aligned}\right\}\tag{2.6}$$

where f, c_φ, c_ψ are constants: m and n are whole numbers which show the number of half-waves of the buckled plate.

Substituting the expressions for w, φ, ψ from (2.6) into (2.5) the following system of algebraic equations is obtained for the three undetermined constants:

$$\left.\begin{aligned}
&\pi^2 p\left(\frac{m^2}{a^2}+\lambda\frac{n^2}{b^2}\right)f-\frac{h^3}{12}\frac{m\pi}{a}c_\varphi-\frac{h^3}{12}\frac{n\pi}{b}c_\psi=0,\\
&\frac{m\pi^3}{a}\left[B_{11}\frac{m^2}{a^2}+(B_{12}+2B_{66})\frac{n^2}{b^2}\right]f-\\
&\quad-\left[a_{55}\frac{\pi^2h^2}{10}\left(B_{11}\frac{m^2}{a^2}+B_{66}\frac{n^2}{b^2}\right)+1\right]c_\varphi-\\
&\quad-a_{44}\frac{\pi^2h^2}{10}(B_{12}+B_{66})\frac{mn}{ab}c_\psi=0,\\
&\frac{n\pi^3}{b}\left[B_{22}\frac{n^2}{b^2}+(B_{12}+2B_{66})\frac{m^2}{a^2}\right]f-a_{55}\frac{\pi^2h^2}{10}(B_{12}+B_{66})\frac{mn}{ab}c_\varphi-\\
&\quad-\left[a_{44}\frac{\pi^2h^2}{10}\left(B_{22}\frac{n^2}{b^2}+B_{66}\frac{m^2}{a^2}\right)+1\right]c_\psi=0.
\end{aligned}\right\}\qquad(2.7)$$

When the determinant of this system is set equal to zero, we obtain an expression for the determination of the critical force p*

$$p^*_{mn}=p^0_{mn}(1-d),\qquad(2.8)$$

where

$$p^0_{mn}=\frac{\pi^2\left[D_{11}\frac{m^4}{a^4}+2(D_{12}+2D_{66})\frac{m^2n^2}{a^2b^2}+D_{22}\frac{n^4}{b^4}\right]}{\frac{m^2}{a^2}+\lambda\frac{n^2}{b^2}},\qquad(2.9)$$

represents the value of the critical load for the plate under consideration as found from the classical theory of orthotropic plates [14]; d is the correction to the classical theory which is determined from (1.21). The constants f, c_φ and c_ψ remain undetermined in the present linear presentation of the problem.

As was mentioned in Section 1 above, d is always smaller than unity. Consequently, the critical forces p^*_{mn} which are determined from (2.8) will be smaller than the corresponding critical forces found with the classical theory. Evidently, from all possible values of p^*_{mn}, the smallest should be selected, and this will be critical. It is easy to see from (2.8) that the value of the critical load depends on many independent geometric and physical quantities of the plate and a complete analysis of the critical stability behavior is difficult in a general case.

Consider the case when a plate is compressed only in the x direction, i.e., $p_1 = p$, $p_2 = 0$. In this case the critical force is determined from (2.8) and (2.9) with $\lambda = 0$. From the above and using (1.21) for d the following expression is obtained with which to determine the critical load:

$$p^*_{mn} = \frac{\pi^2 a^2}{m^2}\left[D_{11}\frac{m^4}{a^4} + 2(D_{12} + 2D_{66})\frac{m^2 n^2}{a^2 b^2} + D_{22}\frac{n^4}{b^4}\right]\frac{1+A}{1+B}, \quad (2.10)$$

where A and B represent the correction when the transverse shear is included. A and B are determined from (1.21).

Thus, formula (2.10) can produce the values of p^*_{mn}, which correspond to all possible combinations of m and n (m = 1, 2, 3, . . . ; n = 1, 2, 3, . . .). As was mentioned before, from all these possible values of p^*_{mn}, the lowest should be selected. Values of p^*_{mn} become lowest, obviously, when only one sinusoidal half-wave is formed in the y direction (which is perpendicular to the direction of compression) i.e., when n = 1. Now we have only to determine that value of m at which $p_{m,1}$ (below the subscript 1 is omitted) obtains its lowest value.

Some numerical results are presented below which were obtained from formula (2.10). For this purpose some of the plate parameters should be defined concisely.

These notations are used

$$\frac{E_1}{E_2} = \frac{\nu_1}{\nu_2} = k_1, \qquad \frac{E_1}{G_{12}} = k_2, \qquad c = \frac{a}{b}, \quad (2.11)$$

as well as the following, which simplifies calculations and decreases the number of parameters:

$$\frac{E_1}{G_{13}} = \frac{E_2}{G_{23}} = k_3. \quad (2.12)$$

Using notations (2.11) and (2.12) we can obtain from (2.10) an expression for the dimensionless quantity $p^*_m\, b^2 / \pi^2 D_{11}$ which would depend on the parameters k_1, k_2, k_3, on the ratio c = a/b, and on the number of half-waves m, n. The expression obtained is very cumbersome and is not given here, although it is useful in calculations. With the use of this formula (in fact (2.10)) values of $\Phi_m(c) = p^*_m\, b^2 / \pi^2\, D_{11}$ have been calculated as a function of c = a/b at m = n = 1, h/b = 0.1 for several values of the parameters k_i. The results are presented in four Tables presented below. In these Tables the values of $\Phi(c)$ found from the classical theory are shown in the columns k_3 = 0, while the last two lines present values of c = a/b and corresponding minimum values of Φ.

$k_1 = k_2 = 5{,}0, \quad \nu_1 = 0{,}3$

c	$k_3 = 0{,}0$	$k_3 = 2{,}0$	$k_3 = 5{,}0$	$k_3 = 10{,}0$
0,35	9,093	7,682	6,241	4,763
0,65	3,357	3,144	2,873	2,516
0,95	2,194	2,105	1,984	1,816
1,25	1,858	1,799	1,716	1,596
1,65	1,817	1,769	1,700	1,598
1,95	1,929	1,881	1,814	1,712
2,25	2,116	2,066	1,995	1,889
2,55	2,360	2,306	2,230	2,115
c	1,5954	1,4820	1,4629	1,4336
$\Phi_{\min}$	1,8000	1,7488	1,6770	1,5694

$k_1 = k_2 = 2{,}0, \ \nu_1 = 0{,}3$

c	$k_3 = 0{,}0$	$k_3 = 2{,}0$	$k_3 = 5{,}0$	$k_3 = 10{,}0$
0,25	18,241	13,263	9,440	6,392
0,45	7,250	6,335	5,335	4,234
0,75	4,269	3,978	3,610	3,130
0,95	3,769	3,569	3,305	2,944
1,25	3,631	3,481	3,277	2,986
1,45	3,737	3,599	3,411	3,137
1,75	4,068	3,936	3,754	3,485
2,15	4,738	4,601	4,411	4,126
c	1,1892	1,1588	1,1151	1,0475
$\Phi_{\min}$	3,6241	3,4657	3,2469	2,9264

$k_1 = 1/2, \ k_2 = 1{,}0, \ \nu_2 = 0{,}3$

c	$k_3 = 0{,}0$	$k_3 = 2{,}0$	$k_3 = 5{,}0$	$k_3 = 10{,}0$
0,25	20,545	14,915	10,570	7,115
0,35	12,828	10,576	8,371	6,212
0,55	8,331	7,519	6,561	5,412
0,75	7,323	6,826	6,196	5,372
1,05	7,532	7,167	6,683	6,008
1,25	8,185	7,844	7,384	6,728
1,45	9,101	8,763	8,301	7,634
1,55	9,641	9,300	8,832	8,151
c	0,8409	0,8036	0,7489	0,6605
$\Phi_{\min}$	7,2484	6,8026	6,1964	5,3267

$k_1 = 1/5$, $k_2 = 1{,}0$, $\nu_2 = 0{,}3$

c	$k_3 = 0{,}0$	$k_3 = 2{,}0$	$k_3 = 5{,}0$	$k_3 = 10{,}0$
0,25	20,840	15,547	11,291	7,784
0,35	13,304	11,239	9,135	6,987
0,55	9,346	8,601	7,691	6,552
0,65	9,007	8,435	7,708	6,751
0,85	9,525	9,087	8,507	7,696
0,95	10,149	9,734	9,177	8,385
1,15	11,897	11,489	10,931	10,119
1,25	12,981	12,565	11,991	11,150
c	0,6688	0,6386	0,5937	0,5177
Φ_{min}	9,0001	8,4329	7,6577	6,5377

We see from these Tables that the difference between the values of critical loads as found from the improved and from the classical theories increases with an increase in k_3. The critical force (2.8) found from the theory presented here is always smaller than the corresponding critical force determined with the classical theory.

In the particular case when the plate is made of transversely isotropic material in such a manner that the plane of isotropy is parallel to the plate middle plane, considering (1.22) and assuming that n = 1, from (2.10) for critical forces we have

$$p_m^* = \frac{\pi^2 D}{b^2} \frac{\left(\frac{m}{c} + \frac{c}{m}\right)^2}{1 + h^* \left(1 + \frac{m^2}{c^2}\right)}, \tag{2.13}$$

where

$$h^* = \frac{\pi^2 h^2}{10 b^2} \frac{E}{(1-\nu^2) G'} = \frac{\pi^2 h^2}{5(1-\nu) b^2} \frac{G}{G'},$$
$$c = \frac{a}{b}, \quad D = \frac{E h^3}{12(1-\nu^2)}. \tag{2.14}$$

Formula (2.13) with h* = 0 produces, evidently, the value of the critical load as found with the classical representation of the problem. It should be noted here, however, that the classical theory, since it is indifferent to the ratio G/G′, produces identical values of critical loads for both isotropic and transversely isotropic plates, provided the modulus of elasticity within the plane of isotropy in the latter case coincides with the modulus of elasticity of the isotropic plate. This indicates clearly the

shortcomings of the classical theory.

Let us now analyze formula (2.13). For this purpose consider the following expression which is obtained directly from (2.13):

$$\Phi_m(c) = \frac{p_m^* b^2}{\pi^2 D} = \frac{\left(\frac{m}{c} + \frac{c}{m}\right)^2}{1 + h^*\left(1 + \frac{m^2}{c^2}\right)} \tag{2.15}$$

It follows from this that at

$$c = m\sqrt{(1 - h^*)/(1 + h^*)} \tag{2.16}$$

the function $\Phi_m(c)$ takes its minimum value

$$\Phi_{\min} = \frac{4}{(1 + h^*)^2}, \tag{2.17}$$

which depends only on the relative plate thickness h^*.

With formulas (2.16) and (2.17) it is easy to find the minimum value of the critical load from (2.13) or (2.15)

$$p_{\min} = \frac{\pi^2 D}{b^2} \frac{4}{(1 + h^*)^2}. \tag{2.18}$$

Just as in the case of classical theory, the minimum value of the critical load does not depend on the number of half-waves of the curved plate. However, it exhibits an extensive dependence on h*.

Without giving elementary details let us plot the stability curves for the transversely isotropic plate for several values of h*. For this purpose we construct a table of Φ(c) values with m = 1 for four values of h*.

c \ h*	0,00	0,0528	0,1053	0,2106
0,4	8.410	6,087	4,770	3,329
0,5	6.250	4,948	4,095	3,045
0,6	5,138	4,286	3,676	2,862
0,7	4,531	3,906	3,432	2,762
0,8	4,203	3,703	3,310	2,730
0,9	4.045	3,619	3,274	2,750
1,0	4.000	3,619	3,304	2,815
1,1	4.036	3,682	3,385	2,915
1,2	4.134	3,796	3,510	
1,3	4,282	3,951		
1,4	4.470			

On the basis of the data in this table it is not difficult to plot stability curves for all values of m = 1, 2, 3, . . . From (2.15) we see that the structure of the function $\Phi_m(c)$ is such that when we have a curve at m = 1 then it is easy to plot curves at m = 2, 3, . . . For this purpose it is sufficient to keep the ordinates of the curve m = 1 (Φ_1) unchanged and increase the abscissas by 2, 3, . . . times, respectively. Curves plotted in such a way are shown in Figure 21. From (2.13), (2.15), the table above and these curves (Figure 21), we see that the critical load is a function of h*.

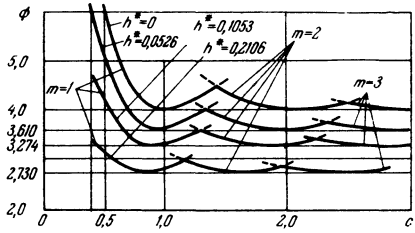

Figure 21.

2. Stability of rectangular transversely isotropic plates with two opposite sides simply-supported. The equations of stability for the transversely isotropic plate in the coordinate system x, y, z, because of (3.2.25), (3.2.26) and (2.3), can be presented as follows

$$\left.\begin{aligned} D\Delta\Delta w &= \left(T_x^0 \frac{\partial^2}{\partial x^2} + T_y^0 \frac{\partial^2}{\partial y^2} + 2S^0 \frac{\partial^2}{\partial x\, \partial y}\right)(w - k\Delta w), \\ \Delta\Phi - \delta^2\Phi &= 0, \end{aligned}\right\} \quad (2.19)$$

where, as before:

$$k = \left(2\frac{G}{G'} - \nu'\frac{E}{E'}\right)\frac{h^2}{10(1-\nu)}, \qquad \delta^2 = \frac{10}{h^2}\frac{G'}{G},$$
$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)},$$

And for internal moment and transverse stress resultants, on the basis of (3.2.15), (3.2.27) and (3.2.28), we have

$$\left.\begin{aligned}
M_x &= -D\left[\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)-\frac{2}{\delta^2}\frac{\partial^2}{\partial y^2}(\Delta w)\right]+\\
&\quad+\frac{2}{\delta^2}\frac{\partial^2\Phi}{\partial x\,\partial y}-k\left(Z-\frac{2}{\delta^2}\frac{\partial^2 Z}{\partial y^2}\right),\\
M_y &= -D\left[\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)-\frac{2}{\delta^2}\frac{\partial^2}{\partial x^2}(\Delta w)\right]-\\
&\quad-\frac{2}{\delta^2}\frac{\partial^2\Phi}{\partial x\,\partial y}-k\left(Z-\frac{2}{\delta^2}\frac{\partial^2 Z}{\partial x^2}\right),\\
H &= -D\left[(1-\nu)\frac{\partial^2 w}{\partial x\,\partial y}+\frac{2}{\delta^2}\frac{\partial^2}{\partial x\,\partial y}(\Delta w)\right]+\Phi-\\
&\quad-\frac{2}{\delta^2}\frac{\partial^2\Phi}{\partial x^2}-\frac{2k}{\delta^2}\frac{\partial^2 Z}{\partial x\,\partial y},\\
N_x &= -D\frac{\partial}{\partial x}(\Delta w)+\frac{\partial\Phi}{\partial y}-k\frac{\partial Z}{\partial x},\\
N_y &= -D\frac{\partial}{\partial y}(\Delta w)-\frac{\partial\Phi}{\partial x}-k\frac{\partial Z}{\partial y}.
\end{aligned}\right\} \qquad (2.20)$$

Let the plate have simply-supported sides (x = 0, x = a) and be compressed by a uniformly distributed load of intensity p (Figure 22). Then using the initial plane stress state, we have for internal inplane stress resultants:

$$T_x^0 = -p, \qquad T_y^0 = S^0 = 0. \qquad (2.21)$$

Substituting T_x^0, T_y^0 and S^0 from (2.21) into (2.19) we obtain the following stability equations for the plate under consideration [15].

$$\Delta\Delta w+\frac{p}{D}\left[\frac{\partial^2 w}{\partial x^2}-k\frac{\partial^2}{\partial x^2}(\Delta w)\right]=0, \quad \Delta\Phi-\delta^2\Phi=0. \qquad (2.22)$$

The boundary conditions of simple support on the two opposite sides x = 0 and x = a are

at x = 0, x = a
$w = 0, M_x = 0, \psi = 0$ (2.23)

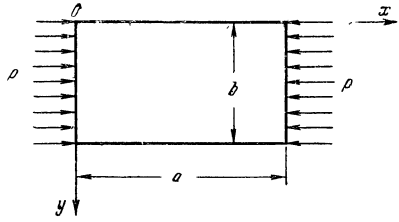

Figure 22.

The solution of (2.22) which satisfies the boundary conditions (2.23) is

$$\left.\begin{aligned} w &= W(y)\sin\frac{m\pi x}{a}, \\ \Phi &= F(y)\cos\frac{m\pi x}{a}, \end{aligned}\right\} \tag{2.24}$$

where m is the number of half-waves of the buckled shape along Ox.

Substituting (2.24) into (2.22) a system of ordinary differential equations with respect to W(y) and F(y) is obtained:

$$\left.\begin{aligned} &\frac{d^4W}{dy^4} - 2\lambda^2\left(1 - \frac{kP}{2D}\right)\frac{d^2W}{dy^2} + \left[\lambda^4 - \frac{p\lambda^2}{D}(1 + k\lambda^2)\right]W = 0, \\ &\frac{d^2F}{dy^2} - (\lambda^2 + \delta^2)F = 0, \qquad \lambda = \frac{m\pi}{a}. \end{aligned}\right\} \tag{2.25}$$

A general solution of (2.25) is

$$W = c_1 \operatorname{sh}\alpha y + c_2 \operatorname{ch}\alpha y + c_3 \sin\beta y + c_4 \cos\beta y, \tag{2.26}$$

$$F = c_5 \operatorname{sh}\omega y + c_6 \operatorname{ch}\omega y, \tag{2.27}$$

where c_i represents the integration constants which are determined from the boundary conditions along the plate sides y = 0 and y = b; and

$$\left.\begin{aligned} \alpha &= \left[\sqrt{\frac{p\lambda^2}{D}\left(1 + \frac{k^2p\lambda^2}{4D}\right)} + \lambda^2\left(1 - \frac{kp}{2D}\right)\right]^{1/2}, \\ \beta &= \left[\sqrt{\frac{p\lambda^2}{D}\left(1 + \frac{k^2p\lambda^2}{4D}\right)} - \lambda^2\left(1 - \frac{kp}{2D}\right)\right]^{1/2}, \end{aligned}\right\} \tag{2.28}$$

$$\omega = \sqrt{\lambda^2 + \delta^2}. \tag{2.29}$$

As was mentioned before, the integration constants c_i, which are present in (2.26) and (2.27), are determined from the boundary conditions along the sides y = 0 and y = b. For example, consider the stability of a plate when the side y = 0 is simply-supported, and the side y = b is completely free [16].

The boundary conditions in this case are

$$\left.\begin{array}{ll} \text{at } y = 0 & \\ & w = 0, \quad M_y = 0, \quad \varphi = 0; \\ \text{at } y = b & \\ & M_y = 0, \quad N_2 = 0, \quad H = 0. \end{array}\right\} \tag{2.30}$$

When the above boundary conditions are satisfied, a uniform system of six algebraic equations with respect to the six integration constants c_i is obtained. When the determinant of this system is set equal to zero the following transcendental equation is obtained from which the critical buckling load may be determined [16]:

$$\alpha_0\left(\beta_0^2+\nu\frac{m^2}{c^2}\right)\left[(1-\nu)\frac{m^2}{c^2}-\left(1+\frac{2m^2}{\delta_0^2 c^2}\right)\left(\alpha_0^2-\frac{m^2}{c^2}\right)\right]\operatorname{tg}\pi\beta_0+$$

$$+\beta_0\left(\alpha_0^2-\nu\frac{m^2}{c^2}\right)\left[(1-\nu)\frac{m^2}{c^2}+\left(1+\frac{2m^2}{\delta_0^2 c^2}\right)\left(\beta_0^2+\frac{m^2}{c^2}\right)\right]\operatorname{th}\pi\alpha_0-$$

$$-2(1-\nu)\alpha_0\beta_0(\alpha_0^2+\beta_0^2)\frac{m^2\omega_0}{c^2\delta_0^2}\operatorname{th}\pi\omega_0=0, \tag{2.31}$$

where we have the notation

$$c=\frac{a}{b}, \quad \alpha_0=\frac{\alpha b}{\pi}, \quad \beta_0=\frac{\beta b}{\pi}, \quad \omega_0=\frac{\omega b}{\pi}, \quad \delta_0=\frac{\delta b}{\pi}, \tag{2.32}$$

as well as the dimensionless terms

$$\eta=\frac{pb^2}{\pi^2 D}, \quad k_0=\frac{\pi^2 k}{2b^2} \tag{2.33}$$

In equation (2.31) α_0 and β_0 are unknown. With the use of formulas

$$\left.\begin{aligned} \alpha_0&=\left[\sqrt{\eta\frac{m^2}{c^2}\left(1+\eta\frac{m^2k_0}{c^2}\right)}+\frac{m^2}{c^2}(1-k_0\eta)\right]^{1/2}, \\ \beta_0&=\left[\sqrt{\eta\frac{m^2}{c^2}\left(1+\eta\frac{m^2k_0}{c^2}\right)}-\frac{m^2}{c^2}(1-k_0\eta)\right]^{1/2} \end{aligned}\right\} \tag{2.34}$$

these terms (α_0 and β_0) are expressed in terms of the parameter η which involves the compressive load p.

Knowing the ratio of the elastic constants of the material and the relative plate dimensions it is possible to find from (2.31) and (2.34) the corresponding value of η for each fixed value of m, and consequently the value of the critical force:

$$p^* = \eta \frac{\pi^2 D}{b^2}. \tag{2.35}$$

From these results we see that by passing to the limit ($k = 0$, $\delta_0 \to \infty$) it is easy to obtain the corresponding results of the classical theory.

When the only phenomena which are considered are related to transverse shear (i.e., when $G/G' \neq 0$, $E/E' = 0$) there is the following interdependence between the parameters k_0 and δ_0:

$$k_0 = \frac{1}{\delta_0^2 (1 - \nu)}. \tag{2.36}$$

On the basis of (2.36) equation (2.31) can be presented in a simplified form.

$$\alpha_0 \left(\beta_0^2 + \nu \frac{m^2}{c^2}\right)^2 \frac{\alpha_0^2 - \frac{m^2}{c^2}}{\beta_0^2 + \frac{m^2}{c^2}} \operatorname{tg} \pi\beta_0 - \beta_0 \left(\alpha_0^2 - \nu \frac{m^2}{c^2}\right)^2 \frac{\beta_0^2 + \frac{m^2}{c^2}}{\alpha_0^2 - \frac{m^2}{c^2}} \operatorname{tg} \pi\alpha_0 +$$

$$+ 2(1 - \nu)\alpha_0\beta_0(\alpha_0^2 + \beta_0^2) \frac{m^2\omega_0}{c^2\delta_0^2} \operatorname{th} \pi\omega_0 = 0. \tag{2.37}$$

It is of importance to note that the smallest critical load in this problem, which corresponds to the smallest positive root of equation (2.31) or (2.37), just as in the classical theory, takes place [17] at $m = 1$, i.e., when buckling of the plate occurs into one half-wave.

Using the above equations, values of η have been calculated for different ratios of the plate sides $c = a/b$ at different ratios of E/G', E/E' and for different values of the ratio h/b for $m = 1$, $\nu = \nu' = 0.25$.

These results [18] show that the value of η (and also the critical load) decreases, for fixed values of the ratio a/b, when the ratio E/G' increases.

		$h/b = 1/10$		$h/b = 1,5$	
E/G' / a/b	0	2,5	5,0	2,5	5,0
0,8	1,989	1,887	1,804	1,660	1,436
1,0	1,434	1,378	1,333	1,255	1,128
1,4	0,952	0,923	0,903	0,868	0,810
2,0	0,698	0,678	0,667	0,648	0,618
2,5	0,610	0,593	0,584	0,570	0,547
3,0	0,563	0,547	0,539	0,527	0,507
4,0	0,516	0,501	0,494	0,483	0,467
5,0	0,495	0,479	0,473	0,463	0,449

		$h/b=1/10$		$h/b = 1/5$	
a/b	$E/G' = 0$ $E/E' = 0$	$E/G' = 2,5$ $E/E' = 1,0$	$E/G' = 5,0$ $E/E' = 2,0$	$E/G' = 2,5$ $E/E' = 1,0$	$E/G' = 5,0$ $E/E' = 2,0$
0,8	1,989	1,896	1,820	1,687	1,476
1,0	1,434	1,382	1,341	1,269	1,150
1,4	0,952	0,925	0,906	0,874	0,819
2,0	0,698	0,679	0,668	0,650	0,621
2,5	0,610	0,594	0,585	0,571	0,549
3,0	0,563	0,547	0,539	0,527	0,509
4,0	0,516	0,501	0,494	0,484	0,468
5,0	0,495	0,480	0,473	0,463	0,449

Comparing the corresponding columns we see that the correction introduced by the consideration of normal stress σ_z is insignificant as compared with the correction which occurs when transverse shear is considered.

3. Stability of transversely isotropic circular plates [19]. Consider the stability of a circular plate of radius a which is compressed along the circumference by a uniformly distributed radial force of intensity p (Figure 23).

In this case we obtain (using the plane stress initial state):

$$T_r^0 = T_\theta^0 = -p, \qquad S = 0. \tag{2.38}$$

Considering (2.1) and (2.38) the following system of differential equations for the stability of a transversely isotropic plate in the coordinate system (z, r, θ) is obtained from (3.7.1) and (3.7.2):

$$\left.\begin{aligned} \Delta\Delta w + \mu^2 \Delta w &= 0, \\ \Delta\Phi - \delta^2\Phi &= 0. \end{aligned}\right\} \tag{2.39}$$

Here

$$\Delta = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial}{\partial \vartheta}\left(\frac{1}{r}\frac{\partial}{\partial \vartheta}\right)\right],$$

$$\delta^2 = \frac{10}{h^2}\frac{G'}{G}, \qquad \mu^2 = \frac{p}{D - kp},$$

$$k = \left(2\frac{G}{G'} - \nu'\frac{E}{E'}\right)\frac{h^2}{10(1-\nu^2)},$$

$$D = \frac{Eh^3}{12(1-\nu^2)}.$$

The solution of (2.39) is sought in the form:

$$\left.\begin{aligned} w(r, \vartheta) &= W(r)\cos n\vartheta, \\ \Phi(r, \vartheta) &= F(r)\sin n\vartheta, \end{aligned}\right\} \tag{2.40}$$

where n = 0, 1, 2, ... is the number of waves of the plate middle surface in a circular direction.

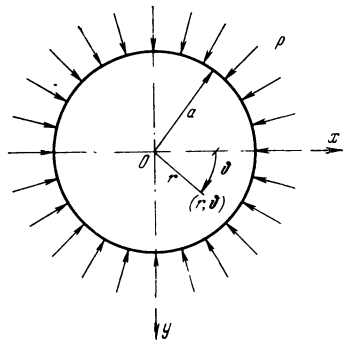

Figure 23.

Substituting w and Φ from (2.40) into (2.39) the following ordinary differential equations with respect to W(r) and F(r) are obtained:

$$\Delta_n \Delta_n W + \mu^2 \Delta_n W = 0, \quad \Delta_n F - \delta^2 F = 0, \tag{2.41}$$

where

$$\Delta_n = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}. \tag{2.42}$$

A general solution of (2.41) is

$$\left.\begin{aligned} W(r) &= c_1 J_n(\mu r) + c_2 Y_n(\mu r) + c_3 r^n + c_4 r^{-n}, \\ F(r) &= c_5 I_n(\delta r) + c_6 K_n(\delta r), \end{aligned}\right\} \tag{2.43}$$

where J_n, Y_n and I_n, K_n are Bessel functions of real and purely imaginary arguments; c_i represent integration constants.

Due to the fact that a full circular plate is considered it should be assumed that $c_2 = c_4 = c_6 = 0$ in (2.43). Then because of (2.40) and (2.43)

$$w = [c_1 Y_n(\mu r) + c_3 r^n] \cos n\vartheta, \tag{2.44}$$

$$\Phi = c_5 I_n(\delta r) \sin n\vartheta. \tag{2.45}$$

The integration constants c_1, c_3 and c_5 present in (2.44) and (2.45) are determined from conditions at the circumference r = a.

Two ways of supporting the circumference are considered below.

Case I. The plate is simply-supported along the circumference. The boundary conditions of the simple supports are

at r = a

$$w = 0, \qquad M_r = 0, \qquad \psi = 0. \tag{2.46}$$

A uniform system of algebraic equations with respect to c_1, c_3 and c_5 can be obtained from (3.7.7), (3.7.8), (2.44) and (2.45) and from the boundary conditions (2.46). When the determinant of this system is set equal to zero the following transcendental equation is obtained, from which the critical load can be determined:

$$\left[1 + \frac{2n}{\delta_0^2} \frac{I_{n-1}(\delta_0)}{I'_n(\delta_0)}\right] \mu J_n(\mu_0) - (1 - \nu)(1 + k_0 \mu_0^2) J_{n+1}(\mu_0) - \\ - \frac{2\mu_0^2}{\delta_0^2} J_{n-1}(\mu_0) = 0, \tag{2.47}$$

where

$$\left.\begin{aligned} \mu_0 = \mu a, \qquad \delta_0 = \delta a, \qquad k_0 = \frac{k}{a^2}, \\ I'_n(x) = \frac{dI_n}{dx} = I_{n-1}(x) - \frac{n}{x} I_n(x). \end{aligned}\right\} \tag{2.48}$$

The critical load p_n^* for each value of n (n = 0, 1, 2, . . .) is determined with the following formula, according to the above

$$p_n^* = \frac{D}{a^2} \frac{\mu_{0n}^2}{1 + k_0 \mu_{0n}^2}, \tag{2.49}$$

where μ_{0n} is the smallest positive root of equation (2.47) for a fixed value of n.

By setting $k_0 = 0, \delta_0 \to \infty$ it is easy to obtain the corresponding equations with which to determine the critical load according to the classical theory.

Case II. The plate is rigidly fixed along the circumference. The boundary conditions in this case are

at r = a

$$\left.\begin{aligned} w &= 0, \\ -\frac{\partial w}{\partial r} + \frac{1}{2G'}\left(\frac{h^2}{4} - \frac{z_0^2}{3}\right)\varphi &= 0, \\ -\frac{1}{r}\frac{\partial w}{\partial \vartheta} + \frac{1}{2G'}\left(\frac{h^2}{4} - \frac{z_0^2}{3}\right)\psi &= 0. \end{aligned}\right\} \tag{2.50}$$

Here, just as in the general theory, it is assumed that the zero conditions of inplane displacements u_r and u_θ are fulfilled only along two circumferences $z = \pm z_0$ $(0 \leqslant z \leqslant h/2)$ on the plate lateral surface.

Using the same approach as in the case of a simply-supported plate the following transcendental equation is obtained for this case

$$J_{n+1}(\mu_0) + \frac{10}{1-\nu}\frac{\mu_0}{(1+k_0\mu_0^2)\delta_0^2}\left(\frac{1}{4} - \frac{1}{3}\frac{z_0^2}{h^2}\right)\left[\mu_0 J_{n-1}(\mu_0) - \right.$$
$$\left. - \frac{nI_{n-1}(\delta_0)}{I'_n(\delta_0)} J_n(\mu_0)\right] = 0. \tag{2.51}$$

The previous notations are used here.

The critical force p_n^* for each value of n is determined from (2.49), where μ_{on} is the smallest positive root of equation (2.51) for a given value of n.

When $k_0 = 0, \delta_0 \to \infty$, the corresponding equation of the classical theory is obtained:

$$J_{n+1}(\mu_0) = 0. \tag{2.52}$$

The critical force in this case is determined from (2.49) with $k_0 = 0$ and the value of the smallest positive root present in (2.49) must be found from (2.52).

It is of interest to note that in the case of the axisymmetric buckled shape (i.e., at n = 0) the instability determined with equations (2.51) and (2.52) are identical. And the difference in critical loads as obtained from the improved and the classical theories is the result of the correction present in (2.49).

Equations (2.47), (2.51) and (2.49) can be used in calculations of critical loads for different plate parameters and physical and mechanical properties.

3. Problems of the Dynamic Stability of Plates

The dynamic stability of orthotropic and transversely isotropic plates are analyzed in this section including the effects of transverse shear.

General statements and definitions on which the dynamic stability of an elastic system is based can be found in a monograph by V. V. Bolotin [20]. This monograph has been used in writing the introductory part of this section.

Dynamic stability problems of plates are located at the junction of two theories, the theory of vibrations and the theory of static stability of plates.

Let the plate be loaded by alternating forces of low magnitude which act in the middle plane. In determining ratios between the frequency of appliedload θ and the frequency of the transverse vibrations ω, the flat shape of the plate becomes dynamically unstable, i.e., transverse vibrations of the plate take place with amplitudes which reach unbounded values.

The alternating load, which may lead to the loss of dynamic stability, is present as one of the parameters in the equation of perturbed equilibrium. Due to this fact such a load is called the parametric. Thus, the theory of the dynamic stability of plates involves the study of transverse vibrations which originate due to the alternating parametric load acting in the middle plane of the plate.

Just as in the case of static stability, we assume that the initial state of stress in problems of dynamic stability involves zero moment. Then we arrive at the concept of a fictitious transverse load

$$\bar{Z} = -\frac{\gamma h}{g}\frac{\partial^2 w}{\partial t^2} + T_x^0\frac{\partial^2 w}{\partial x^2} + T_y^0\frac{\partial^2 w}{\partial y^2} + 2S^0\frac{\partial^2 w}{\partial x\,\partial y}, \qquad (3.1)$$

where the first term represents the normal component of the inertia force (the inplane components of the inertial forces are omitted here);

$$T_x^0 = T_x^0(x, y, t), \quad T_y^0 = T_y^0(x, y, t), \quad S^0 = S^0(x, y, t)$$

are inplane stress resultants of the initial flat state of the plate, and generally speaking these stress resultants are periodic functions of time. In the problems considered below the inplane components of the fictitious load are neglected.

With this fictitious load and the system of differential equations of plate equilibrium (Section 2 of this Chapter), a system of differential equations for the dynamic stability of a plate can be derived.

Dynamic stability of a simply-supported (along the entire contour) rectangular orthotropic plate [21]. Consider the dynamic stability of a simply-supported rectangular (a x b) orthotropic plate which is loaded by a compressive periodic

load acting in the principal direction x. Let the intensity of this uniformly distributed load be p(t). Then because of the flat initial state the following expression is obtained for the inplane stress resultants:

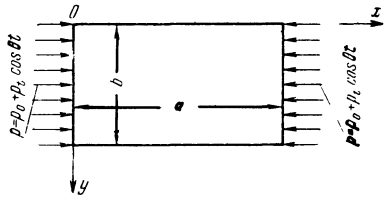

Figure 24.

$$T_x^0 = -p(t), \quad T_y^0 = 0, \quad S^0 = 0, \tag{3.2}$$

and for the fictitious transverse load we have

$$\tilde{Z} = -\frac{\gamma h}{g}\frac{\partial^2 w}{\partial t^2} - p(t)\frac{\partial^2 w}{\partial x^2}. \tag{3.3}$$

Substituting $\tilde{Z}$ from (3.3) in place of Z_2 in equations (2.5.25) – (2.5.27), the following system of differential equations for the dynamic stability of an orthotropic plate is obtained, including the effects of transverse shear:

$$\left.\begin{aligned}
&\frac{h^3}{12}\left(\frac{\partial \varphi}{\partial x}+\frac{\partial \psi}{\partial y}\right)-p(t)\frac{\partial^2 w}{\partial x^2}-\frac{\gamma h}{g}\frac{\partial^2 w}{\partial t^2}=0,\\
&\left[B_{11}\frac{\partial^3}{\partial x^3}+(B_{12}+2B_{66})\frac{\partial^3}{\partial x\,\partial y^2}\right]w-\\
&-\frac{h^2}{10}\left[a_{55}\left(B_{11}\frac{\partial^2}{\partial x^2}+B_{66}\frac{\partial^2}{\partial y^2}\right)\varphi+a_{44}(B_{12}+B_{66})\frac{\partial^2 \psi}{\partial x\,\partial y}\right]+\varphi=0,\\
&\left[B_{22}\frac{\partial^3}{\partial y^3}+(B_{12}+2B_{66})\frac{\partial^3}{\partial y\,\partial x^2}\right]w-\\
&-\frac{h^2}{10}\left[a_{44}\left(B_{22}\frac{\partial^2}{\partial y^2}+B_{66}\frac{\partial^2}{\partial x^2}\right)\psi+a_{55}(B_{12}+B_{66})\frac{\partial^2 \varphi}{\partial y\,\partial x}\right]+\psi=0
\end{aligned}\right\} \tag{3.4}$$

The boundary conditions of simple supports should be added to equations (3.4):

at x = 0, x = a w = 0, M_x = 0, ψ = 0;

at y = 0, y = b w = 0, M_y = 0, φ = 0.

The solution of (3.4) is sought in the form

$$\left.\begin{aligned} w(x,y,t) &= f_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},\\ \varphi(x,y,t) &= \varphi_{mn}(t)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b},\\ \psi(x,y,t) &= \psi_{mn}(t)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}, \end{aligned}\right\} \tag{3.5}$$

where $f_{mn}(t)$, $\varphi_{mn}(t)$ and $\psi_{mn}(t)$ are some functions of time, which represent the values of the functions w, φ, ψ at the plate center at any time; m and n are whole numbers. It is possible to satisfy all the conditions of the simple supports by representing the function w, φ and ψ in the form (3.5).

Substituting (3.5) into (3.4) we obtain

$$\frac{\pi^2 m}{a^2}p(t)-\frac{h^3}{12}\frac{\pi m}{q}\varphi_{mn}(t)-\frac{h^3}{12}\frac{\pi n}{b}\psi_{mn}(t)=\frac{\gamma h}{g}\frac{d^2 f_{mn}(t)}{dt^2}, \tag{3.6}$$

$$\frac{m\pi^3}{a}\left[B_{11}\frac{m^2}{a^2}+(B_{12}+2B_{66})\frac{n^2}{b^2}\right]f_{mn}(t)-$$

$$-\left[a_{55}\frac{\pi^2 h^2}{10}\left(B_{11}\frac{m^2}{a^2}+B_{66}\frac{n^2}{b^2}\right)+1\right]\varphi_{mn}(t)-$$

$$-a_{44}\frac{\pi^2 h^2}{10}(B_{12}+B_{66})\frac{mn}{ab}\psi_{mn}(t)=0, \tag{3.7}$$

$$\frac{n\pi^3}{b}\left[B_{22}\frac{n^2}{b^2}+(B_{12}+2B_{66})\frac{m^2}{a^2}\right]f_{mn}(t)-$$

$$-\left[a_{44}\frac{\pi^2 h^2}{10}\left(B_{22}\frac{n^2}{b^2}+B_{66}\frac{m^2}{a^2}\right)+1\right]\psi_{mn}(t)-$$

$$-a_{55}\frac{\pi^2 h^2}{10}(B_{12}+B_{66})\frac{mn}{ab}\varphi_{mn}(t)=0. \tag{3.8}$$

When the functions $\varphi_{mn}(t)$ and $\psi_{mn}(t)$ are eliminated from (3.6) with the use of (3.7) and (3.8), the following ordinary differential equation of the second order with respect to $f_{mn}(t)$ is obtained:

$$\frac{d^2 f_{m,n}(t)}{dt^2}+\omega_{mn}^2\left[1-\frac{p(t)}{p^*_{mn}}\right]f_{mn}(t)=0; \tag{3.9}$$

where ω_{mn} is the frequency of natural vibration of the plate (1.19); p_{mn} is the value of the critical force when the plate becomes unstable (2.10).

In the general case when p(t) is a periodic function of the type

$$p(t) = p_0 + p_t\Phi(t),$$

where Φ(t) is a periodic function with period T:

$$\Phi(t + T) = \Phi(t),$$

then equation (3.9) is called the Hill equation [22, 23].

In our case when p(t) is a periodic function of a particular type, we have

$$p(t) = p_0 + p_t \cos \vartheta t, \tag{3.10}$$

where θ is the frequency of the compressive load; p(t) is the magnitude of the exciting force. Equation (3.9) can then be presented in the following form after some elementary transformations:

$$\frac{d^2 f_{mn}}{dt^2} + \Omega_{mn}^2 (1 - 2\mu_{mn} \cos \vartheta t) f = 0, \tag{3.11}$$

where Ω_{mn} is the frequency of natural vibration of the plate when loaded with a constant component of the external load (p_0) and is determined from

$$\Omega_{mn} = \omega_{mn} \sqrt{1 - \frac{p_0}{p^*_{mn}}}; \tag{3.12}$$

and μ_{mn} is a coefficient which is called the excitation coefficient and is of the form:

$$\mu_{mn} = \frac{p_t}{2(p^*_{mn} - p_0)}. \tag{3.13}$$

Equation (3.11) is the Mathieu equation, which has successively increasing solutions for certain ratios of the coefficients. These solutions occupy a considerable region of the parameters and represent the region of the dynamic instability of the plate.

The boundaries of the first three regions of dynamic instability for this problem can be determined with the following approximate equations [24].

The first (principal) instability area:

$$\vartheta^* = 2\Omega_{mn} \sqrt{1 \pm \mu_{mn}}; \tag{3.14}$$

The second instability area:

$$\left.\begin{aligned} \vartheta^* &= \Omega_{mn}\sqrt{1+\frac{1}{3}\mu_{mn}^2}, \\ \vartheta^* &= \Omega_{mn}\sqrt{1-2\mu_{mn}^2}; \end{aligned}\right\} \tag{3.15}$$

The third instability area:

$$\vartheta^* = \frac{2}{3}\Omega_{mn}\sqrt{1-\frac{9\mu^2_{mn}}{8\pm 9\mu_{mn}}}. \tag{3.16}$$

These equations establish the relationship between the frequency of the compressive force and that of the natural vibration of the plate in the vicinity of the regions in which the dynamic instabilities are located. The critical frequencies of the external force are represented here by θ^*, corresponding to the boundaries of the regions of dynamic instability.

From equation (3.13) and taking into account p^*_{mn}, p_0 and p_1 we see that the excitation coefficient μ_{mn} is a positive value and cannot exceed 1/2 because

$$p_0 + p_t \leqslant p^*_{mn}$$

(Otherwise the plate becomes unstable because of the static application of the load $p_0 + p_t$). Considering this and equations (3.14) – (3.16) we see that the width of the region of dynamic instability rapidly decreases as the number of the regions increases. The first principal instability region has the greatest width and from the point of view of practical applications it is of the most interest.

Instability regions are easily plotted in a rectangular system of coordinates where $\theta^*/2\Omega_{mn}$ is plotted on the ordinate and μ_{mn} on the abscissa.

In order to solve the problem of the dynamic instability of a plate it is necessary to find a point within the plane $(\mu_{mn}, \theta^*/2\Omega_{mn})$ which would correspond to a given relationship of the parameters. If this point is outside of the instability region this means that the initial flat shape of the plate is stable. On the other hand, if this point is located inside of any instability region, then any initial transverse deviation of the plate will become limitless with time, i.e., the flat shape of the plate will be dynamically unstable at a given value of the frequency of the external load.

Let us consider a case when the constant intensity component of the external compressive force p_0 is zero [25], i.e.,

$$p(t) = p_t \cos \vartheta t, \tag{3.17}$$

Then equation (3.11) is

$$\frac{d^2 f_{mn}}{dt^2} + \omega_{mn}^2 (1 - 2\mu_{mn} \cos \vartheta t) f_{mn} = 0, \tag{3.18}$$

where

$$\mu_{mn} = \frac{p_t}{2p^*_{mn}}. \tag{3.19}$$

In this case the boundaries of the first three instability regions are obtained from (3.14) – (3.16) as

$$\vartheta^* = 2\omega_{mn}\sqrt{1 \pm \mu_{mn}}; \tag{3.20}$$

$$\vartheta^* = \omega_{mn}\sqrt{1 + \frac{1}{3}\mu_{mn}^2}, \quad \vartheta^* = \omega_{mn}\sqrt{1 - 2\mu^2_{mn}}; \tag{3.21}$$

$$\vartheta^* = \frac{2}{3}\omega_{mn}\sqrt{1 - \frac{9\mu^2_{mn}}{8 \pm 9\mu_{mn}}}. \tag{3.22}$$

Let us rewrite equations (3.20) – (3.22) by considering (1.19) and (2.8). First of all, for the excitation coefficients μ_{mn} because of (2.8) and (3.19) we obtain

$$\left.\begin{aligned} \mu_{mn} &= \frac{p_t}{2p^0_{mn}(1-d)}, \\ \text{or} \\ \mu_{mn} &= \frac{\overset{0}{\mu}_{mn}}{1-d}, \end{aligned}\right\} \tag{3.23}$$

where for the excitation coefficient, μ^0_{mn}, without considering the transverse shear, we would have!

$$\overset{0}{\mu}_{mn} = \frac{p_t}{2p^0_{mn}}. \tag{3.24}$$

Furthermore, considering (1.19) and (3.24) expressions are obtained from (3.20) – (3.22) for the boundaries of the instability regions

$$\vartheta^* = 2\omega^0_{mn}\sqrt{1 - d \pm \mu^0_{mn}}; \tag{3.25}$$

$$\left.\begin{aligned} \vartheta^* &= \omega^0_{mn}\sqrt{1 - d + \frac{1}{3}\frac{(\mu^0_{mn})^2}{1-d}}, \\ \vartheta^* &= \omega^0_{mn}\sqrt{1 - d - 2\frac{(\mu^0_{mn})^2}{1-d}}; \end{aligned}\right\} \tag{3.26}$$

$$\vartheta^* = \frac{2}{3}\omega^0_{mn}\sqrt{1-d-\frac{9(\mu^0_{mn})^2}{8(1-d)\pm 9\mu^0_{mn}}}. \qquad (3.27)$$

Thus, with equations (3.25) – (3.27) the boundaries of the first three instability regions of the orthotropic plate can be determined. A distinctive characteristic of these equations is the fact that they are constructed in terms of the results of the classical theory of plates ω^0_{mn}, μ^0_{mn} with the correction d due to transverse shear (1.21) taken into account. Evidently when d = 0 the results will correspond to those of the classical theory.

Considering that $\mu_{mn} \leqslant 1/2$ it is easy to find from (3.23) the limiting value of the excitation coefficient μ^0_{mn}:

$$\mu^0_{mn,\,\lim} = \frac{1-d}{2}$$

and this value is evidently lower than that of μ_{mn} for which

$$\mu_{mn,\,\lim} = \frac{1}{2}$$

These results are sufficient for a successful construction of the instability regions of the orthotropic plate under consideration.

The four tables below present the results obtained with formulas (3.25) – (3.28) for a square (a = b) orthotropic plate when only one half-wave is formed in the directions x and y, i.e., when m = 1, n = 1. These tables present the ratio $\theta^*/2\omega^0_{11}$ as a function of the excitation coefficient μ^0_{11} for ν_1 = 0.3 and different values of the k_i ($k_1 = E_1/E_2 = \nu_1/\nu_2$; $k_2 = E_1/G_{12}$; $k_3 = E_1/G_{13} = E_2/G_{23}$)..

$k_1 = 2{,}0,\ k_2 = 2{,}0,\ k_3 = 2{,}0$

μ^0_{11}	Region I		Region II		Region III	
0,0	0,974	0,974	0,487	0,487	0,325	0,325
0,1	0,922	1,024	0,482	0,488	0,322	0,323
0,2	0,866	1,072	0,465	0,491	0,314	0,318
0,3	0,806	1,118	0,436	0,495	0,295	0,311
0,4	0,741	1,162	0,391	0,501	0,256	0,302
0,475	0,689	1,193	0,344	0,507	0,194	0,294

$k_1 = 2{,}0,\ k_2 = 2{,}0,\ k_3 = 5{,}0$

μ_{11}^0	Region I		Region II		Region III	
0,0	0,939	0,939	0,470	0,470	0,313	0,313
0,1	0,885	0,991	0,464	0,471	0,311	0,311
0,2	0,826	1,040	0,445	0,474	0,301	0,306
0,3	0,763	1,087	0,412	0,479	0,278	0,298
0,4	0,695	1,132	0,360	0,486	0,228	0,288
0,441	0,664	1,150	0,332	0,489	0,187	0,284

$k_1 = 5{,}0,\ k_2 = 5{,}0,\ k_3 = 2{,}0$

μ_{11}^0	Region I		Region II		Region III	
0,0	0,980	0,980	0,490	0,490	0,327	0,327
0,1	0,928	1,030	0,485	0,491	0,325	0,325
0,2	0,872	1,077	0,468	0,494	0,316	0,320
0,3	0,813	1,123	0,440	0,498	0,298	0,313
0,4	0,749	1,167	0,396	0,504	0,260	0,304
0,481	0,693	1,201	0,347	0,510	0,195	0,296

$k_1 = 5{,}0,\ k_2 = 5{,}0,\ k_3 = 5{,}0$

μ_{11}^0	Region I		Region II		Region III	
0,0	0,953	0,953	0,477	0,477	0,318	0,318
0,1	0,899	1,004	0,471	0,478	0,315	0,316
0,2	0,842	1,053	0,453	0,480	0,306	0,310
0,3	0,780	1,099	0,421	0,485	0,285	0,303
0,4	0,713	1,144	0,373	0,492	0,239	0,294
0,454	0,674	1,167	0,337	0,496	0,190	0,288

The last lines of these tables give the highest and lowest values of the ratio $\theta^*/2\omega^0_{11}$ for all three instability regions at the corresponding limiting values of $\mu^0_{11,\text{lim}}$ (which are presented at the end of the first column in each table). Using these numerical results it is easy to construct the first three instability regions for the dynamic stability problem for certain plate materials.

Consider now the case when the plate is made of transversely isotropic material in such a manner that the plane of isotropy at each point is parallel to the middle plane of the plate. Then because of (1.22) we obtain from (1.21)

$$1 - d = \left[1 + h^*\left(\frac{m^2}{c^2} + n^2\right)\right]^{-1}, \tag{3.30}$$

where h* and c are defined in (2.14).

From (3.25) – (3.28) and (3.30) it is not difficult to derive equations which can be used to determine the boundaries of the dynamic instability regions of a transversely isotropic plate. In particular, for a square plate (a = b, c = 1) when only one half-wave is formed in the x and y directions, i.e., when m = 1 and n = 1, the following expressions for the ratio $\theta^*/2\omega^0$ of each of the first three instability regions are obtained from (3.25) – (3.27) (subscripts "11" are omitted):

$$\frac{\vartheta^*}{2\omega^0} = \sqrt{\frac{1}{1+2h^*} \pm \mu^0}, \tag{3.31}$$

$$\left.\begin{aligned} \frac{\vartheta^*}{2\omega^0} &= \frac{1}{2}\sqrt{\frac{1}{1+2h^*} + \frac{1+2h^*}{3}(\mu^0)^2}, \\ \frac{\vartheta^*}{2\omega^0} &= \frac{1}{2}\sqrt{\frac{1}{1+2h^*} - 2(1+2h^*)(\mu^0)^2}; \end{aligned}\right\} \tag{3.32}$$

$$\frac{\vartheta^*}{2\omega^0} = \frac{1}{3}\sqrt{\frac{1}{1+2h^*} - \frac{9(1+2h^*)(\mu^0)^2}{8 \pm 9(1+2h^*)\mu^0}}. \tag{3.33}$$

and from (3.24) and (3.28) we have

$$\mu^0 = \frac{p_t}{2p^0} \qquad \mu^0_{\text{пр}} = \frac{1}{2(1+2h^*)} \tag{3.34}$$

Assuming that h* = 0 in these formulas the known results of the classical theory are obtained.

Results obtained from (3.31) – (3.34) are given in the three tables below. Values of $\theta^*/2\omega^0$ are given as functions of μ^0 for certain values of h*.

$h^* = 0$

μ^0	Region I		Region II		Region III	
0,0	1,00	1,00	0,50	0,50	0,333	0,333
0,10	0,949	1,049	0,495	0,501	0,331	0,332
0,20	0,894	1,095	0,479	0,503	0,324	0,327
0,30	0,837	1,140	0,453	0,507	0,307	0,321
0,40	0,775	1,183	0,412	0,513	0,273	0,312
0,50	0,707	1,225	0,354	0,520	0,199	0,302

$h^* = 0,10$

μ^0	Region I		Region II		Region III	
0,0	0,913	0,913	0,456	0,456	0,304	0,304
0,10	0,856	0,966	0,450	0,458	0,301	0,302
0,20	0,796	1,017	0,429	0,461	0,291	0,296
0,30	0,730	1,065	0,393	0,467	0,264	0,288
0,40	0,658	1,111	0,335	0,474	0,201	0,278
5/12	0,616	1,118	0,308	0,475	0,182	0,276

$h^* = 0,20$

μ^0	Region I		Region II		Region III	
0,0	0,845	0,845	0,423	0,423	0,282	0,282
0,10	0,784	0,902	0,414	0,424	0,278	0,279
0,20	0,717	0,956	0,388	0,428	0,263	0,272
0,30	0,644	1,007	0,340	0,435	0,223	0,262
5/14	0,598	1,035	0,299	0,440	0,168	0,255

The last lines of these tables give the limiting values of μ^0_{lim} and the corresponding values of $\theta^*/2\omega^0$.

The first Table corresponds to the classical theory. Figure 25 shows the regions of dynamic instability of the plate under consideration at h* = 0.0 and h* = 0.2. These areas were constructed on the basis of the numerical results given in the first and the third tables.

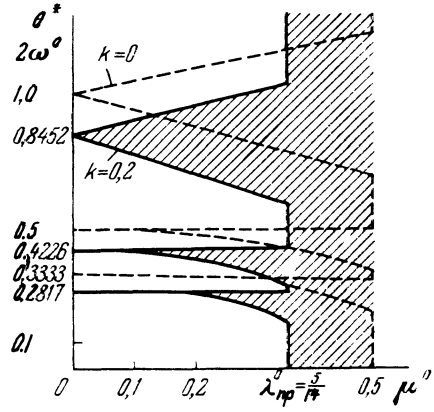

Figure 25.

Examination of Figure 25 and the tables and equations of this Section reveals that with an increase in the parameter k_3 or h* the instability regions calculated from (3.25) – (3.27) or from (3.31) – (3.33) differ more and more from the corresponding regions as determined with the classical theory ($k_3 = 0$ or $h^* = 0$). This difference is explained by the fact that an increase in k_3 or h* leads to a decrease in $\theta^*/2\omega^0$ and of μ^0. In fact we find here a narrowing of the plate instability region.

4. Vibration and Stability of Plates in the Presence of Large Displacements

The problems of free and forced vibrations and of static and dynamic instability of a rectangular (a x b) orthotropic plate, which is simply-supported along the entire contour, are considered in this Section including the possibility of large displacements (Figure 13 and 20). The influence of the normal stress σ_z, rotational inertia, and inplane components of the inertial force are disregarded.

Similar problems for a three-layer plate have been studied before [26]. The material in this section is presented as in this previous work [26].

1. Post-critical state and parametric vibration of plates. Let the plate be subjected to the action of uniformly distributed compressive forces of intensity $p_1(t)$ and $p_2(t)$ within the xy plane (Figure 26).

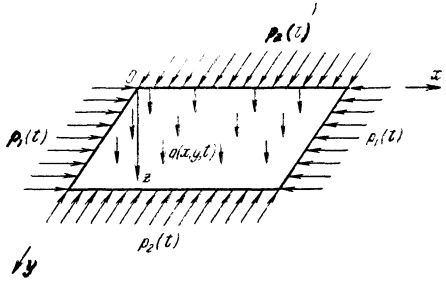

Figure 26.

Furthermore, a linear decay of the plate vibration takes place with a decay coefficient ϵ. Then, considering an initial flat state for the plate, we obtain for the "fictitious" transverse load (see equations (2.1), (2.3), (3.2) and (3.3)):

$$\tilde{Z} = -\rho h \frac{\partial^2 w}{\partial t^2} + T_x^0 \frac{\partial^2 w}{\partial x^2} + T_y^0 \frac{\partial^2 w}{\partial y^2} - 2\rho h \varepsilon \frac{\partial w}{\partial t}. \tag{4.1}$$

where ρ is the material density; $\rho = \gamma/g$, $T_x^0 = -p_1(t)$, $T_y^0 = -p_2(t)$.

The governing system of differential equations is derived from (2.8.29) – (2.8.32) with $Z_2 = \tilde{Z}$, $A_1 = A_2 = 0$:

$$\frac{h^3}{12}\left(\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y}\right) + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - 2\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2 F}{\partial x\,\partial y} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2} - p_1(t)\frac{\partial^2 w}{\partial x^2} - p_2(t)\frac{\partial^2 w}{\partial y^2} - ph\frac{\partial^2 w}{\partial t^2} - 2ph\varepsilon\frac{\partial w}{\partial t} = 0, \tag{4.2}$$

$$B_{11}\frac{\partial^3 w}{\partial x^3} + (B_{12} + 2B_{66})\frac{\partial^3 w}{\partial x\,\partial y^2} - \frac{h^2}{10}\left[\frac{B_{11}}{B_{55}}\frac{\partial^2 \varphi}{\partial x^2} + \frac{B_{66}}{B_{55}}\frac{\partial^2 \varphi}{\partial y^2} + \frac{B_{12} + B_{66}}{B_{44}}\frac{\partial^2 \psi}{\partial x\,\partial y}\right] + \varphi = 0, \tag{4.3}$$

$$B_{22}\frac{\partial^3 w}{\partial y^3} + (B_{12} + 2B_{66})\frac{\partial^3 w}{\partial y\,\partial x^2} - \frac{h^2}{10}\left[\frac{B_{22}}{B_{44}}\frac{\partial^2 \psi}{\partial y^2} + \frac{B_{66}}{B_{44}}\frac{\partial^2 \psi}{\partial x^2} + \frac{B_{12} + B_{66}}{B_{55}}\frac{\partial^2 \varphi}{\partial y\,\partial x}\right] + \psi = 0, \tag{4.4}$$

$$\frac{a_{22}}{h}\frac{\partial^4 F}{\partial x^4}+\frac{a_{66}+2a_{12}}{h}\frac{\partial^4 F}{\partial x^2\,\partial y^2}+\frac{a_{11}}{h}\frac{\partial^4 F}{\partial y^4}-$$

$$-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}=0. \qquad (4.5)$$

We let the stress function F(x, y, t), normal displacement w(x, y, t) and the transverse shear functions φ(x, y, t), ψ(x, y, t) be represented as

$$\left.\begin{aligned} F &= F_{mn}(t)\,\sin\lambda_n x\,\sin\mu_m y,\\ w &= f_{mn}(t)\,\sin\lambda_n x\,\sin\mu_m y,\\ \varphi &= \varphi_{mn}(t)\,\cos\lambda_n x\,\sin\mu_m y,\\ \psi &= \psi_{mn}(t)\,\sin\lambda_n x\,\cos\mu_m y, \end{aligned}\right\} \qquad (4.6)$$

where

$$\lambda_n=\frac{n\pi}{a}, \qquad \mu_m=\frac{m\pi}{b}, \qquad (4.7)$$

are undetermined functions of time.

The conditions of simple supports, which can be presented as follows, are identically satisfied:

at x = 0, x = a

$$w=0, \quad M_x=0, \quad \psi=0, \quad \frac{\partial^2 F}{\partial y^2}=0 \quad \int_0^b \frac{\partial^2 F}{\partial x\,\partial y}\,dy=0;$$

at y = 0, y = b

$$w=0, \quad M_y=0, \quad \varphi=0, \quad \frac{\partial^2 F}{\partial x^2}=0, \quad \int_0^a \frac{\partial^2 F}{\partial x\,\partial y}\,dx=0.$$

Substituting the expressions for the functions w, φ, ψ from (2.6) into (4.2) and (4.3) the functions $\varphi_{mn}(t)$ and $\psi_{mn}(t)$ can be easily represented in terms of the function $f_{mn}(t)$:

$$\varphi_{mn}=K_{1mn}f_{mn}(t), \qquad \psi_{mn}=K_{2mn}f_{mn}(t), \qquad (4.8)$$

where

$$K_{1mn}=\frac{Q_1R_{22}-Q_2R_{12}}{R_{11}R_{22}-R_{12}R_{21}}, \quad K_{2mn}=\frac{Q_2R_{11}-Q_1R_{21}}{R_{11}R_{22}-R_{12}R_{21}}$$

$$Q_1=D_{11}\lambda_n+(D_{12}+2D_{66})\,\lambda_n\mu_m^2, \quad Q_2=D_{22}\mu_m^3+(D_{12}+2D_{66})\,\lambda_n^2\mu_m,$$

$$R_{11}=\frac{h^5}{120}\left(\frac{B_{11}}{B_{55}}\lambda_n^2+\frac{B_{66}}{B_{55}}\mu_m^2\right)+\frac{h^3}{12},$$

$$R_{22} = \frac{h^5}{120}\left(\frac{B_{22}}{B_{44}}\mu_m^2 + \frac{B_{66}}{B_{44}}\lambda_n^2\right) + \frac{h^3}{12},$$

$$R_{12} = \frac{h^5}{120}\frac{B_{12}+B_{66}}{B}\lambda_n\mu_m, \quad R_{21} = \frac{h^5}{120}\frac{B_{12}+B_{66}}{B_{55}}\lambda_n\mu_m.$$

When the expressions for w and F from (4.6) are substituted into (4.5) and the method of Bubnov-Galerkin is applied, then for $F_{mn}(t)$ we have

$$F_{mn} = -\frac{16h}{3ab}\frac{\lambda_n\mu_m}{a_{22}\lambda_n^4 + (a_{66}+2a_{12})\lambda_n^2\mu_m^2 + a_{11}\mu_m^4} f_{mn}^2. \tag{4.9}$$

Thus, using the last three equations of the governing system of differential equations we have represented F_{mn}, φ_{mn} and ψ_{mn} with the use of equations (4.8) and (4.9), in terms of the function f(t).

Only one equation, (4.2), and the one undetermined function f(t) remain. Equation (4.2) is solved by the Bubnov-Galerkin method. In the process of deriving an equation of the type (3.9.11) where

$$L(x,y) = \frac{h^3}{12}\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}\right) + \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2} -$$
$$- 2\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2 F}{\partial x\,\partial y} - p_1(t)\frac{\partial^2 w}{\partial x^2} - p_2(t)\frac{\partial^2 w}{\partial y^2} - \rho h\frac{\partial^2 w}{\partial t^2} + 2\rho h\varepsilon\frac{\partial w}{\partial t},$$

and equations (4.6), (4.8), (4.9) are available for φ, ψ, F and w, the following nonlinear differential equation with variable coefficients is obtained in terms of $f_{mn}(t)$:

$$f''_{mn} + 2\varepsilon f'_{mn} + \omega_{mn}^2\left[1 - \frac{p_1(t)}{p^*_{1mn}} - \frac{p_2(t)}{p^*_{2mn}}\right]f_{mn} + \alpha_{mn}f_{mn}^3 = 0, \tag{4.10}$$

where

$$\omega_{mn}^2 = \frac{\Phi_{mn}}{\rho h}, \quad p^*_{1mn} = \frac{\Phi_{mn}}{\lambda_n^2}, \quad p^*_{2mn} = \frac{\Phi_{mn}}{\mu_m^2}, \tag{4.11}$$

$$\alpha_{mn} = \frac{512}{9\rho a^2 b^2}\frac{\lambda_n^2\mu_m^2}{a_{22}\lambda_n^4 + (a_{66}+2a_{12})\lambda_n^2\mu_m^2 + a_{11}\mu_m^4}, \tag{4.12}$$

$$\Phi_{mn} = \frac{h^3}{12}(K_{1mn}\lambda_n + K_{2mn}\mu_m), \quad f'_{mn} = \frac{df}{dt}. \tag{4.13}$$

Comparison of equation (4.11) with (1.19) and (2.8) shows that ω_{mn} is the linear value of the frequency of natural vibration, and p^*_{1mn} and p^*_{2mn} are the critical value of the forces p_1 and p_2 acting independently.

From equation (4.10) it is easy to establish that the correction due to transverse shear in (3.9.13) is present only in the linear term. On the other hand, the coefficient a_{mn} does not depend on the plate transverse shear properties.

When the elements which characterize dynamic processes are eliminated from (4.10) and using the assumption that f_{mn}, p_1 and p_2 do not depend on time, then we have

$$(p - p^*_{mn}) f_{mn} - \overset{0}{a}_{mn} f^3_{mn} = 0, \tag{4.14}$$

which describes the post-critical behavior of a plate subjected to a load of the type $p_1 = p$, $p_2 = kp$. The following notations have been used here:

$$\left.\begin{aligned} p^*_{mn} &= \frac{\Phi_{mn}}{\lambda_n^2 + k\mu_m^2}, \\ \overset{0}{a}_{mn} &= \frac{512h}{9a^2b^2} \frac{\lambda_n^2 \mu_m^2}{(\lambda_n^2 + k\mu_m^2)[a_{22}\lambda_n^4 + (a_{66} + 2a_{12})\lambda_n^2 \mu_m^2 + a_{11}\mu_m^4]}. \end{aligned}\right\} \tag{4.15}$$

If we consider that the plate deflection f_{mn} is not zero after the plate becomes unstable (as a result of the critical load p^*_{mn}), then from (4.14) we have

$$p = p^*_{mn} + \overset{0}{a}_{mn} f^2; \tag{4.16}$$

This expression shows a relationship between the load and the plate deflection in the post-critical state ($p > p^*_{mn}$).

As is known from Section 2 of this Chapter, critical forces p^*_{mn} as determined including the consideration of transverse shear are always smaller than the corresponding critical forces found from the classical theory. On the other hand, the consideration of transverse shear does not introduce any changes into the coefficient of the linear term in (4.14). Thus it is not difficult to see from (4.16) that the post-critical state of the plate, when transverse shear is taken into account, is achieved with a force which is smaller than the corresponding force found from the classical theory.

Assuming that

$$p_i(t) = p_{0i} + p_{ti} \cos \vartheta t \qquad (i = 1, 2), \tag{4.17}$$

we obtain from (4.10) an equation for the nonlinear parametrically excited vibration of the plate

$$f''_{mn} + 2\varepsilon f'_{mn} + \Omega^2_{mn}(1 - 2\mu_{mn}\cos\vartheta t) f_{mn} + \alpha_{mn} f^3_{mn} = 0, \quad (4.18)$$

where

$$\Omega^2_{mn} = \omega^2_{mn}\left(1 - \frac{p_{01}p^*_{mn2} + p_{02}p^*_{mn1}}{p^*_{mn1}p^*_{mn2}}\right), \quad (4.19)$$

$$\mu_{mn} = \frac{1}{2}\,\frac{p_{1t}p^*_{mn2} + p_{2t}p^*_{mn1}}{p^*_{mn1}p^*_{mn2} - p_{01}p^*_{mn2} - p_{02}p^*_{mn1}} \quad (4.20)$$

are the frequency Ω_{mn} and excitation coefficient μ_{mn} of the plate subjected to constant external forces acting within the midplane. Equation (4.18) in terms of its harmonic approximation is [27]:

$$f(t) = a\sin\frac{\vartheta t}{2} + b\cos\frac{\vartheta t}{2}; \quad (4.21)$$

It is known that three solutions of this type exist. One of them corresponds to the flat plate shape, the second is unstable, and the third one (steady-state bending vibrations) is

$$a = \frac{2\Omega_{mn}}{\sqrt{3\alpha_{mn}}}\sqrt{\frac{\vartheta^2}{4\Omega^2_{mn}} - 1 \pm \sqrt{\mu^2_{mn} - \frac{\vartheta^2\varepsilon^2}{\Omega^4_{mn}}}}, \quad b = 0. \quad (4.22)$$

Because of (4.22) we obtain the amplitude of vibration in the presence of the principal parametric resonance

$$C^2_{mn} = \frac{4\Omega^2_{mn}}{3\alpha_{mn}}\left(\frac{\vartheta^2}{4\Omega^2_{mn}} - 1 \pm \sqrt{\mu^2_{mn} - \frac{\vartheta^2\varepsilon^2}{\Omega^4_{mn}}}\right). \quad (4.23)$$

Boundaries of the regions of the principal parametric resonance, i.e., the first region of dynamic instability, are determined in the usual way with the use of (3.14), which in this case is

$$(\vartheta^*_{mn})^2 = 4\Omega^2_{mn}\left(1 \mp \sqrt{\mu^2_{mn} - \frac{4\varepsilon^2}{\Omega^2_{mn}}}\right). \quad (4.24)$$

In the case of a conservative problem ($\epsilon = 0$) from (4.23) and (4.24) we obtain

$$C_{mn} = \frac{1}{3\alpha_{mn}}[\vartheta^2 - (\vartheta^*_{mn})^2], \quad (\vartheta_{mn})^2 = 4\Omega^2_{mn}(1 \mp \mu_{mn}). \quad (4.25)$$

respectively.

Thus, the consideration of the effect of transverse shear, while lowering considerably the frequency of natural vibration, does not change the value of the coefficient a_{mn}. Therefore, the value of the critical frequencies of the principal parametric resonance decrease, and the amplitude-frequency characteristics $c_{mn}^2 - \theta^2$ displace in a parallel manner to lower frequencies.

A numerical example is given below. Let the transversely isotropic square plate (a = b), for which $h^* = 0.024\pi^2$ (see (1.25)), be loaded in such a way that $p_{01} = p_{02} = p_{t2} = 0$, $p_{t1} = 0.4422\, p^*_{mn1}$ and let it be vibrated only by identical sinusoidal half-waves (m = n = 1).

On the basis of the above, from (4.19) and (4.20) while considering (1.23), (1.24), (2.13) and (3.24), we obtain for Ω^2 and μ (indexes m = 1 and n = 1 are omitted):

$$\begin{aligned} (\Omega^0)^2 &= (\omega^0)^2, & \mu^0 &= 0{,}1500, \\ \Omega^2 &= 0{,}6785\,(\omega^0)^2, & \mu &= 0{,}2211, \end{aligned}$$

where just as before, the values found with the classical theory are designated by zeros. Furthermore, by designating with "+" and "–" subscripts the upper and lower boundaries of the instability regions, we have from (4.25)

$$\begin{aligned} (\vartheta_-^0)^2 &= 3{,}400\,(\omega^0)^2, \quad (\vartheta_+^0)^2 = 4{,}600\,(\omega^0)^2, \\ (\vartheta_-^*)^2 &= 2{,}114\,(\omega^0)^2, \quad (\vartheta_+^*)^2 = 3{,}314\,(\omega^0)^2. \end{aligned}$$

The first line represents frequencies as found from the classical theory.

Amplitudes of the established resonance vibrations within the region of the principal parametric resonance from (4.25) are

$$\left.\begin{aligned} 3\alpha\,(C_-^0)^2 &= \vartheta^2 - 3{,}400\,(\omega^0)^2, \\ 3\alpha\,(C_+^0) &= \vartheta^2 - 4{,}600\,(\omega^0)^2 \end{aligned}\right\} \quad \text{classical theory}$$

$$\left.\begin{aligned} 3\alpha\,(C_-)^2 &= \vartheta^2 - 2{,}114\,(\omega^0)^2, \\ 3\alpha\,(C_+)^2 &= \vartheta^2 - 3{,}314\,(\omega^0)^2 \end{aligned}\right\} \quad \text{this theory}$$

This example illustrates sufficiently the effect of transverse shear.

2. Problems of transverse vibrations. Let the plate be subjected to the action of a normally applied load distributed according to q(x, y, t) (Figure 26). Let the vibrations follow a linear decay with decay coefficient ϵ. Then the governing system of differential equations for the nonlinear vibrations of the orthotropic plate can be written:

$$\frac{h^3}{12}\left(\frac{\partial\varphi}{\partial x}+\frac{\partial\psi}{\partial y}\right)+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 F}{\partial y^2}-2\frac{\partial^2 w}{\partial x\,\partial y}\frac{\partial^2 F}{\partial x\,\partial y}+\frac{\partial^2 w}{\partial y^2}\frac{\partial^2 F}{\partial x^2}-$$
$$-\rho h\frac{\partial^2 w}{\partial t^2}-2\rho h\varepsilon\frac{\partial w}{\partial t}=-q(x,y,t), \quad (4.26)$$

$$B_{11}\frac{\partial^3 w}{\partial x^3}+(B_{12}+2B_{66})\frac{\partial^3 w}{\partial x\partial y^2}-\frac{h^2}{10}\left[\frac{B_{11}}{B_{55}}\frac{\partial^2\varphi}{\partial x^2}+\right.$$
$$\left.+\frac{B_{66}}{B_{55}}\frac{\partial^2\varphi}{\partial y^2}+\frac{B_{12}+B_{66}}{B_{44}}\frac{\partial^2\psi}{\partial x\,\partial y}\right]+\varphi=0, \quad (4.27)$$

$$B_{22}\frac{\partial^3 w}{\partial y^3}+(B_{12}+2B_{66})\frac{\partial^3 w}{\partial y\,\partial x^2}-\frac{h^2}{10}\left[\frac{B_{22}}{B_{44}}\frac{\partial^2\psi}{\partial y^2}+\right.$$
$$\left.+\frac{B_{66}}{B_{44}}\frac{\partial^2\psi}{\partial x^2}+\frac{B_{12}+B_{66}}{B_{55}}\frac{\partial^2\varphi}{\partial x\,\partial y}\right]+\psi=0, \quad (4.28)$$

$$a_{22}\frac{\partial^4 F}{\partial x^4}+\frac{a_{66}+2a_{12}}{h}\frac{\partial^4 F}{\partial x^2\partial y^2}+\frac{a_{11}}{h}\frac{\partial^4 F}{\partial y^4}-$$
$$-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2+\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}=0. \quad (4.29)$$

When the functions F(x, y, t), w(x, y, t), φ(x, y, t) and ψ(x, y, t) are assumed in the form (4.6), then the conditions of simple supports will be identically satisfied. By following the same principle as in the previous problems, relationships for the functions F_{mn}, φ_{mn} and ψ_{mn} in the form of equations (4.9) and (4.8) are found, and for determining $f_{mn}(t)$, the following differential equation is obtained

$$f''_{mn}+2\varepsilon f'_{mn}+\omega^2_{mn}f_{mn}+\alpha_{mn}f^3_{mn}=q_{mn}(t), \quad (4.30)$$

where along with (4.11) – (4.13) we have

$$q_{mn}(t)=\frac{4}{ab\rho h}\int_0^a\int_0^b q(x,y,t)\sin\lambda_n x\sin\mu_m y\,dx\,dy. \quad (4.31)$$

When $q_{mn} = 0$ and $\epsilon = 0$ in (4.30), an equation for the undamped natural nonlinear vibration of the plate is derived

$$f''_{mn}+\omega^2_{mn}f_{mn}+\alpha_{mn}f^3_{mn}=0. \quad (4.32)$$

Assuming that

$$f_{mn}=C_{mn}\cos\tilde{\omega}_{mn}t, \quad (4.33)$$

where $\tilde{\omega}_{mn}$ is the vibration frequency, and C_{mn} is the amplitude, then for the frequency of the natural nonlinear vibration in the first approximation [28] we have

$$\tilde{\omega}_{mn}^2 = \omega_{mn}^2 + \frac{3}{4}\alpha_{mn}C_{mn}^2. \tag{4.34}$$

It is evident from (4.34) that with an increase in vibration amplitude (C_{mn}) the frequency of nonlinear vibration increases. On the other hand, as was shown before, including the effects of transverse shear leads to a decrease in the frequency of natural linear vibrations (ω_{mn}). It is evident, therefore, that when phenomena related to transverse shear are taken into account the frequency of nonlinear natural vibration could be lower than that determined neglecting the effects of transverse shear (Figure 27).

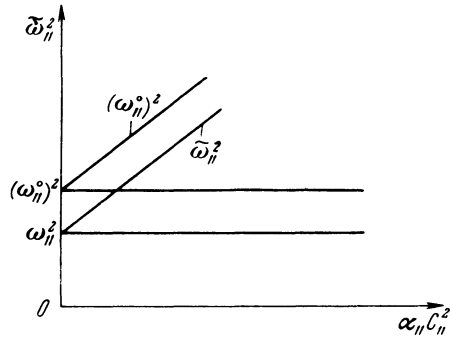

Figure 27.

Figure 27 shows the amplitude-frequency relationship for nonlinear natural vibration of a transversely isotropic square plate (a = b) at m = n = 1 for two values of $h^* = 0.024\pi^2$ and $h^* = 0$, i.e., with and without consideration of transverse shear.

Assuming that $q_{mn}(t) = q_{mn} \cos \theta t$ in (4.30) an equation for forced vibrations is obtained:

$$f''_{mn} + 2\varepsilon f'_{mn} + \omega_{mn}^2 f_{mn} + \alpha_{mn} f_{mn}^3 = q_{mn} \cos \vartheta t. \tag{4.35}$$

Equations of the type (4.35) have been sufficiently studied before [29]. In this book we follow the principles presented in the literature [30].

Assume that we are in the vicinity of the usual resonance, i.e.,

$$\vartheta = \omega_{mn} + \delta_{mn}, \tag{4.36}$$

where δ_{mn} is a small value.

A relationship between the amplitude C_{mn} of forced vibration and the frequency of the external force $q_{mn}(t)$ in the vicinity of the usual resonance is, in terms of a linear approximation,

$$C_{mn}^2(\delta_{mn}^2 + \varepsilon^2) = \frac{q_{mn}^2}{4\omega_{mn}^2}. \tag{4.37}$$

When the nonlinearity is taken into account in the first approximation then (4.34) is obtained and from it we have

$$\tilde{\omega}_{mn} = \omega_{mn} + \frac{3\alpha_{mn}}{8\omega} C_{mn}^2. \tag{4.38}$$

In accordance with this the following equation is obtained when ω_{mn} is replaced with $\tilde{\omega}_{mn}$ in (4.37)

$$C_{mn}^2\left[\left(\delta_{mn} - \frac{3\alpha_{mn}}{8\omega_{mn}} C_{mn}^2\right)^2 + \varepsilon^2\right] = \frac{q_{mn}^2}{4\omega_{mn}^2}. \tag{4.39}$$

Equation (4.39) is cubic with respect to C_{mn}^2. Starting with a definite value of q_{mn}, namely,

$$(q_{mn}^*)^2 = \frac{64\omega_{mn}^3\varepsilon^3}{3\alpha_{mn}}, \tag{4.40}$$

equation (4.39) has three real roots defining a positive frequency region. The boundary of this region is determined by the condition $dC_{mn}/d\delta_{mn} = \infty$, or

$$\delta_{mn}^2 + \varepsilon^2 - \frac{3}{2}\alpha_{mn}\frac{\delta_{mn}}{\omega_{mn}} C_{mn}^2 + \frac{27}{64}\alpha_{mn}^2\frac{1}{\omega_{mn}^2} C_{mn} = 0. \tag{4.41}$$

When equation (4.41) and (4.39) are solved simultaneously, points C and D of the resonant curve (4.39) are found (Figure 28).

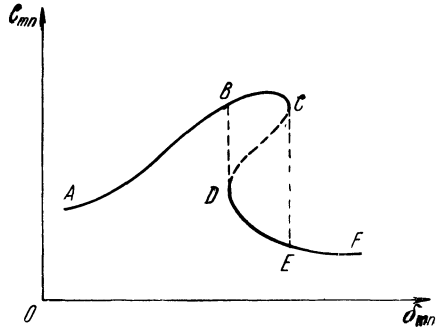

Figure 28.

The section CD corresponds to unstable plate vibrations. At point C the amplitude breaks and it drops to point E. The amplitude jumps to point B during the reverse process.

It should be also noted that the nonlinearity does not influence the maximum value of the amplitude of the established vibrations which is

$$(C_{mn})_{\max} = \frac{q_{mn}}{2\omega_{mn}\varepsilon}. \tag{4.42}$$

Finally, since the frequency of natural linear vibration ω_{mn} decreases when transverse shear is considered, the phenomenon of resonance starts considerably earlier, as compared with the classical case, and the maximum value of the amplitude of established vibration will increase considerably.

5. Stability of Anisotropic Plates Subjected to Supersonic Gas Flow

In this Section the stability problem of an infinite anisotropic plate subjected to supersonic gas flow on one side is considered. The direction of flow is along the x axis with a non-perturbed velocity U.

Let the plate material obey the generalized Hooke's law and have one plane of elastic symmetry which is parallel to the plate middle plane at each plate point.

It is assumed that the gas pressure p on the plate surface can be calculated from the approximate formula of "piston theory"[31].

$$p = p_\infty\left(1 + \frac{\varkappa - 1}{2}\frac{v}{a_\infty}\right)^{\frac{2\varkappa}{\varkappa-1}}, \tag{5.1}$$

where p_∞ is the pressure of non-perturbed gas flow; v is the normal component of velocity of the plate surface; a_∞ is the velocity of sound for non-perturbed flow; $\varkappa$ is the polytropic exponent.

The transverse load Z(x, y, t) (which must be substituted for Z_2) from the governing system of differential equations (2.3.1) – (2.3.5) is found to be composed of the inertial forces, damping forces and aerodynamic pressure and can be presented in the following way [32].

$$Z = -\rho h\frac{\partial^2 w}{\partial t^2} - 2\rho h\varepsilon\frac{\partial w}{\partial t} - \frac{\varkappa p_\infty}{a_\infty}\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}\right), \tag{5.2}$$

where ρ is the material density; ϵ is the damping factor. The last term in this equation represents the additional pressure obtained by expanding (5.1) into a series. It was assumed in this case that

$$v = \frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}. \tag{5.3}$$

The governing system of differential equations for determining the stability of an anisotropic plate which is subjected to supersonic gas flow can be obtained from (2.3.1) – (2.3.5) when $u = v = 0$; $A_1 = A_2 = A_3 = 0$; $X_i = Y_i = 0$; $Z_1 = 0$ and transverse shear of the plate are taken into account. In the derivation of this system of equations (2.4.1), (2.4.2) and (5.2) were considered. After some elementary transformations the following system of differential equations for this problem is obtained:

$$\left.\begin{aligned} \frac{h^3}{12}\left(\frac{\partial \varphi}{\partial x}+\frac{\partial \psi}{\partial y}\right)-\rho h\frac{\partial^2 w}{\partial t^2}-2\rho h\varepsilon\frac{\partial w}{\partial t}-\frac{\varkappa p_\infty}{a_\infty}\left(\frac{\partial w}{\partial t}+U\frac{\partial w}{\partial x}\right)=0,\\ L_{13}(D_{ik})\,w-\frac{h^2}{10}(L_{1\varphi}+L_{12}\psi)+\frac{h^3}{12}\varphi=0,\\ L_{23}(D_{ik})\,w-\frac{h^2}{10}(L_2\psi+L_{21}\varphi)+\frac{h^3}{12}\psi=0, \end{aligned}\right\} \tag{5.4}$$

where (2.3.7) and (2.3.8) define the linear operators L.

Let an infinite number of waves with half-wave lengths λ_x and λ_y be formed in the x and y directions of the plate, respectively.

Then the solution of (5.4) is [33].

$$\left.\begin{aligned} \varphi &= iV_1 e^{i(\omega t-rx-sy)},\\ \psi &= iV_2 e^{i(\omega t-rx-sy)},\\ w &= V_3 e^{i(\omega t-rx-sy)}, \end{aligned}\right\} \tag{5.5}$$

where V_j represent complex coefficients; ω is the number of vibrations of the streamlined plate; $r = \pi/\lambda_x$ and $s = \pi/\lambda_y$ are wave numbers. Here $i = \sqrt{-1}$.

Substituting the expressions for φ, ψ, w from (5.5) into (5.4) a uniform system of three algebraic equations with respect to the three unknowns V_j is obtained. When the determinant of this system is set equal to zero the following characteristic equation is obtained:

$$\omega^2 - i\omega(2\varepsilon + \mu) + ir\mu U - \Omega^2 = 0, \tag{5.6}$$

where

$$\mu = \frac{\varkappa p_\infty}{\rho h a_\infty}, \tag{5.7}$$

$$\Omega^2 = \frac{h^3}{12\rho h}\,\frac{sC_1 - rC_2}{C_3}, \tag{5.8}$$

Also, Ω^2 is the square of the natural frequency of transverse vibration of the plate in a vacuum. Furthermore:

$$C_1 = \frac{h^2}{10}[D_{22}s^3 + 3D_{26}rs^2 + (D_{12} + 2D_{66})r^2s +$$
$$+ D_{16}r^3]\Big\{(a_{55}D_{11} + a_{45}D_{16})r^2 + [a_{45}(D_{12} + D_{66}) + 2a_{55}D_{16}]rs +$$
$$+ (a_{55}D_{66} + a_{45}D_{26})s^2 + \frac{5}{6}h\Big\} - \frac{h^2}{10}[D_{11}r^3 + 3D_{16}r^2s +$$
$$+ (D_{12} + 2D_{66})rs^2 + D_{26}s^3]\{(a_{45}D_{22} + a_{55}D_{26})s^2 +$$
$$+ [a_{55}(D_{12} + D_{66}) + 2a_{45}D_{26}]rs + (a_{45}D_{66} + a_{55}D_{16})r^2\},$$

$$C_2 = \frac{h^2}{10}[D_{22}s^3 + 3D_{26}rs^2 + (D_{12} + 2D_{66})r^2s +$$
$$+ D_{16}r^3]\{(a_{45}D_{11} + a_{44}D_{16})r^2 + [a_{44}(D_{12} + D_{66}) + 2a_{45}D_{16}]rs +$$
$$+ (a_{45}D_{66} + a_{44}D_{26})s^2\} - \frac{h^2}{10}[D_{11}r^3 + 3D_{16}r^2s +$$
$$+ (D_{12} + 2D_{66})rs^2 + D_{26}s^3]\{(a_{44}D_{22} + a_{45}D_{26})s^2 +$$
$$+ [a_{45}(D_{12} + D_{66}) + 2a_{44}D_{26}]rs + (a_{44}D_{66} + a_{45}D_{16})r^2 + \frac{5}{6}h\},$$

$$C_3 = \left(\frac{h^2}{10}\right)^2\Big\{(a_{55}D_{11} + a_{45}D_{16})r^2 + [a_{45}(D_{12} + D_{66}) + 2a_{55}D_{16}]rs +$$
$$+ (a_{55}D_{66} + a_{45}D_{26})s^2 + \frac{5}{6}h\Big\}\Big\{(a_{44}D_{22} + a_{45}D_{26})s^2 +$$
$$+ [a_{45}(D_{12} + D_{66}) + 2a_{44}D_{26}]rs + (a_{44}D_{66} + a_{45}D_{16})r^2 + \frac{5}{6}h\Big\} -$$
$$- \left(\frac{h^2}{10}\right)^2\{(a_{45}D_{11} + a_{44}D_{16})r^2 + [a_{44}(D_{12} + D_{66}) + 2a_{45}D_{16}]rs +$$
$$+ (a_{45}D_{66} + a_{44}D_{26})s^2\}\{(a_{45}D_{22} + a_{55}D_{26})s^2 +$$
$$+ [a_{55}(D_{12} + D_{66}) + 2a_{45}D_{26}]rs + (a_{45}D_{66} + a_{55}D_{16})r^2\}.$$

From equation (5.6) we have for the frequency of vibration of the plate

$$\omega = \frac{1}{2}[i(2\varepsilon + \mu) \pm \sqrt{4\Omega^2 - 4ir\mu U - (2\varepsilon + \mu)^2}]. \qquad (5.9)$$

On the basis of (5.5) it is easy to establish that if the imaginary part of ω, i.e.,

$$\operatorname{Im}\omega = \frac{1}{2}\Bigg[(2\varepsilon + \mu) \pm \sqrt{\frac{4\Omega^2 - (2\varepsilon + \mu)}{2}} \times$$
$$\times \sqrt{\sqrt{1 + \frac{16r^2\mu^2U^2}{[4\Omega^2 - (2\varepsilon + \mu)^2]^2}} - 1}\Bigg], \qquad (5.10)$$

is positive, then non-perturbed motion of the plate is stable for small perturbations. Thus, for the velocity of non-perturbed flow for which the plate motion is stable, we obtain from the condition Im $\omega \geqslant 0$

$$U \leqslant \frac{\Omega}{r}\left(1+\frac{2\varepsilon}{\mu}\right), \tag{5.11}$$

where the second term represents the constructive damping.

For the critical velocity from (5.11) we have

$$U_{cr} = \frac{\Omega}{r}\left(1+\frac{2\varepsilon}{\mu}\right). \tag{5.12}$$

As was mentioned before (Section 1 of this Chapter) consideration of the phenomenon related to transverse shear results in a decrease in the frequency of the free plate vibrations Ω. Therefore, on the basis of (5.12) it can be said that the consideration of transverse shear also decreases the value of the critical velocity U_{cr} at which a loss of stability of the non-perturbed plate motion is possible.

Plates made of orthotropic materials in such a way that the principal elasticity directions of the plate material do not coincide with the principal geometric directions, i.e., do not coincide with directions x = constant and y = constant, are of great interest.

Let the plate be made of orthotropic material in such a manner that one of its planes of elastic symmetry is parallel to the plate middle plane and two other principal directions of elasticity form an angle φ with the principal geometric directions (x and y) at each point of the plate.

Furthermore, let the elastic constants of the material in the principal directions of elasticity, i.e., at $\varphi = 0$, be known and be designated in the usual way but with primes, namely, B'_{jk} and a'_{jk}.

Then on the basis of (1.5.1) – (1.5.9) and (2.2.8) – (2.2.11) the following expressions are obtained for the elastic constants present in the coefficients C_j:

$$\begin{aligned}
B_{11} &= B'_{11}\cos^4\varphi + 2(B'_{12}+2B'_{66})\sin^2\varphi\cos^2\varphi + B'_{22}\sin^4\varphi,\\
B_{22} &= B'_{11}\sin^4\varphi + 2(B'_{12}+2B'_{66})\sin^2\varphi\cos^2\varphi + B'_{22}\cos^4\varphi,\\
B_{12} &= B'_{12} + [B'_{11}+B'_{22}-2(B'_{12}+2B'_{66})]\sin^2\varphi\cos^2\varphi,\\
B_{66} &= B'_{66} + [B'_{11}+B'_{22}-2(B'_{12}+2B'_{66})]\sin^2\varphi\cos^2\varphi,\\
B_{16} &= \frac{1}{2}[B'_{22}\sin^2\varphi - B'_{11}\cos^2\varphi + (B'_{12}+2B_{66})\cos 2\varphi]\sin 2\varphi,\\
B_{26} &= \frac{1}{2}[B'_{22}\cos^2\varphi - B'_{11}\sin^2\varphi - (B'_{12}+2B'_{66})\cos 2\varphi]\sin 2\varphi,
\end{aligned}$$

$$a_{44} = a'_{44}\cos^2\varphi + a'_{55}\sin^2\varphi, \qquad a_{55} = a'_{44}\sin^2\varphi + a'_{55}\cos^2\varphi,$$

$$a_{45} = (a'_{44} - a'_{55})\sin\varphi\cos\varphi.$$

Examination of the expressions for B_{jk} and a_{jk} as well as formulas (5.8) and (5.12) shows that the critical velocity U_{cr} depends considerably on the orientation of the principal elasticity directions of the plate material and is a periodic function of the orientation angle φ.

Those values of the half-wave lengths λ_x and λ_y are of greatest interest in the vicinity of which the critical velocity achieves its minimum value for a given orientation angle φ and given elastic properties of the plate material. These values are determined from the following system of nonlinear algebraic equations:

$$\frac{\partial U_{cr}}{\partial \lambda_x} = 0, \qquad \frac{\partial U_{cr}}{\partial \lambda_y} = 0. \tag{5.13}$$

It is clear that the system of equations (5.13) in the general case is cumbersome and can be solved only by numerical methods.

In order to illustrate the influence of transverse shear and the orientation angle φ the stability solution for the infinite orthotropic plate with noncoinciding principal geometric and elasticity directions is given for the case when the plate is subjected to supersonic gas flow in the direction of the x axis.

Let $B_{22} = 11B_{11} = 11B$; $B_{12} = 0$; $B_{66} = 0.5B$, $a_{55} = 5a_{44} = 5a$; $\lambda_y = \infty$ or $s = 0$.

Three cases of $Ba = k$ are considered, namely, $k = 0$ (classical theory), $k = 8.0$ and $k = 10.0$ (two cases when transverse shear are accounted for) at different orientation angles φ.

Figure 29 shows a graphical representation of the values $U^* = U_{cr}\rho h/Dr^2(1 + 2\epsilon/\mu)^2$.

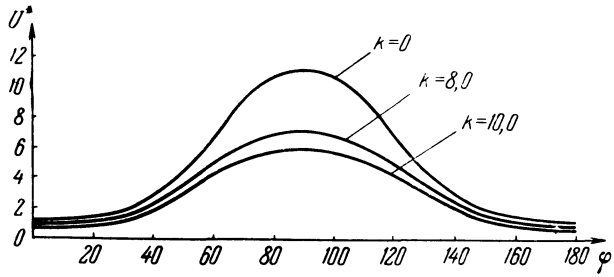

Figure 29.

Examination of the curves in this figure shows that the critical value of the velocity of the gas flow decreases considerably when the transverse shear is considered. However, by varying the orientation of the principal directions of elasticity of the plate material it is possible to increase the critical velocity of the gas flow.

6. Stability of Transversely Isotropic Non-Elastic Plates with Consideration of Transverse Shear [34]

Consider the problem of buckling of a non-elastic plate under conditions of continuous loading when the transverse deformations of the plate are considered. This problem is solved with the theory of plastic flow [35]. Similar problems have been studied before [36] but without the consideration of transverse shear.

While considering the basic principles of the theory of plastic flow, the author's improved theory (at $A_1 = A_2 = 0$) uses the hypothesis of a continuous loading [37–39] according to which the bending of the plate is possible as the load increases insuring loading on all points of the plate.

Let an initially flat state be realized in a plate which undergoes deformation beyond the limits of elasticity:

$$\sigma_x = -p, \qquad \sigma_y = -q, \qquad \tau_{xy} = r. \tag{6.1}$$

During buckling the stresses in a plate increase by negligibly small amounts $\delta\sigma_x$, $\delta\sigma_y$, $\delta\tau_{xy}$, $\delta\tau_{xz}$, $\delta\tau_{yz}$. At the same time the deformation components undergo correspondingly small increases. In the general case for the plate under consideration we have [40]

$$\left.\begin{aligned}
\delta e_x &= \frac{1}{E}(\delta\sigma_x - \nu\delta\sigma_y) - \frac{1}{3}F(T)\,\delta T(2p - q),\\
\delta e_y &= \frac{1}{E}(\delta\sigma_y - \nu\delta\sigma_x) - \frac{1}{3}F(T)\,\delta T(2q - p),\\
\delta e_{xy} &= \frac{1}{G}\delta\tau_{xy} + 2F(T)\,\delta T r,\\
\delta e_{xz} &= \frac{1}{G'}\delta\tau_{xz}, \qquad \delta e_{yz} = \frac{1}{G'}\delta\tau_{yz},
\end{aligned}\right\} \tag{6.2}$$

And also

$$\left.\begin{aligned}
F(T) &= \frac{1}{2T^2}\frac{d\Phi}{dT}, \qquad T^2 = \frac{p^2 - pq + q^2 + 3r^2}{3},\\
\delta T &= -\frac{1}{6T}[(2p - q)\,\delta\sigma_x + (2q - p)\,\delta\sigma_y - 6r\delta\tau_{xy}],
\end{aligned}\right\} \tag{6.3}$$

where $\Phi(T)$ is the stress intensity which is a function of T and which is characteristic of a given material and which does not depend on the type of stress state. On the basis of (6.3) and (6.2) we obtain

$$\left.\begin{aligned} E\delta e_x &= A_{11}\delta\sigma_x + A_{12}\delta\sigma_y + A_{16}\delta\tau_{xy}, \\ E\delta e_y &= A_{22}\delta\sigma_y + A_{12}\delta\sigma_x + A_{26}\delta\tau_{xy}, \\ E\delta e_{xy} &= A_{66}\delta\tau_{xy} + A_{16}\delta\sigma_x + A_{26}\delta\sigma_y, \\ E\delta e_{xz} &= A_{55}\delta\tau_{xz},\ E\delta e_{yz} = A_{44}\delta\tau_{yz}, \end{aligned}\right\} \tag{6.4}$$

where the following notations are used

$$\left.\begin{aligned} &A_{11} = 1 + \vartheta(2p - q)^2, \quad A_{22} = 1 + \vartheta(2q - p)^2 \\ &A_{66} = \frac{E}{G} + 36\vartheta r^2, \quad A_{44} = A_{55} = \frac{E}{G'}, \\ &A_{12} = \vartheta(2q - p)(2p - q) - \nu, \\ &A_{16} = -6\vartheta r(2p - q), \quad A_{26} = -6\vartheta r(2q - p), \end{aligned}\right\} \tag{6.5}$$

$$\vartheta = \frac{E}{18}\frac{F(T)}{T}. \tag{6.6}$$

When (6.4) is solved with respect to the stress increments we obtain

$$\left.\begin{aligned} \delta\sigma_x &= b_{11}\delta e_x + b_{12}\delta e_y + b_{16}\delta e_{xy}, \\ \delta\sigma_y &= b_{22}\delta e_y + b_{12}\delta e_x + b_{26}\delta e_{xy}, \\ \delta\tau_{xy} &= b_{66}\delta e_{xy} + b_{16}\delta e_x + b_{26}\delta e_y, \\ \delta\tau_{xz} &= a_{55}\delta e_{xz},\ \delta\tau_{yz} = a_{44}\delta e_{yz}; \end{aligned}\right\} \tag{6.7}$$

Here

$$\left.\begin{aligned} &b_{11} = \frac{E}{\Omega}(A_{22}A_{66} - A_{26}^2), \quad b_{22} = \frac{E}{\Omega}(A_{11}A_{66} - A_{16}^2), \\ &b_{12} = \frac{E}{\Omega}(A_{16}A_{26} - A_{12}A_{66}), \quad b_{66} = \frac{E}{\Omega}(A_{11}A_{22} - A_{12}^2), \\ &b_{16} = \frac{E}{\Omega}(A_{12}A_{26} - A_{22}A_{16}), \quad b_{26} = \frac{E}{\Omega}(A_{12}A_{16} - A_{11}A_{26}), \\ &a_{55} = G', \quad a_{44} = G', \\ &\Omega = A_{66}\left[A_{11}A_{22} - A_{12}^2 + \frac{2A_{12}A_{16}A_{26} - A_{11}A_{26}^2 - A_{22}A_{16}^2}{A_{66}}\right]. \end{aligned}\right\} \tag{6.8}$$

With the assumption that $X^{\pm} = Y^{\pm} = 0$ and $f_1(z) = f_2(z) = f(z)$ we obtain for shear stresses, from (2.2.2)

$$\tau_{xz} = f(z)\varphi(x, y), \qquad \tau_{yz} = f(z)\psi(x, y).$$

Because of (6.7) we have for transverse strain increments

$$\delta e_{xz} = \frac{1}{a_{55}} f(z)\varphi(x, y), \qquad \delta e_{yz} = \frac{1}{a_{44}} f(z)\psi(x, y), \tag{6.9}$$

where $\varphi(x, y)$ and $\psi(x, y)$ are functions which characterize infinitely small increases of the transverse strains. (Remember that during the initial flat state $\tau_{xz} = \tau_{yz} = 0, \varphi = \psi = 0$.)

From (2.2.18) and (2.2.19), and on the basis of the above conditions, expressions are derived for shear displacements of the buckled plate with the additional assumption that $X_i = Y_i = 0$, $J_{01} = J_{02} = 0$ and $a_{45} = 0$ (transversely isotropic plate):

$$\left.\begin{aligned} u_x &= u^0 + \delta u - z\frac{\partial w}{\partial x} + \frac{J_0}{a_{55}}\varphi, \\ u_y &= v^0 + \delta v - z\frac{\partial w}{\partial y} + \frac{J_0}{a_{44}}\psi, \end{aligned}\right\} \tag{6.10}$$

where $u^0 = u^0(x, y)$; $v^0 = v^0(x, y)$ are inplane displacements of points of the middle plane which correspond to the flat plane state of stress (6.1); $\delta u = \delta u(x, y)$, $\delta v = \delta v(x, y)$, $w = w(x, y)$ are the increases in the inplane and normal displacements of points of the plate middle surface during buckling; $J_0 = \int_0^z f(z)dz$, i.e., defined as in (2.2.20).

On the basis of (6.10) and (1.2.2) for the strain increases e_x, e_y and e_{xy} during buckling we have

$$\left.\begin{aligned} \delta e_x &= \frac{\partial \delta u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{J_0}{a_{55}}\frac{\partial \varphi}{\partial x}, \\ \delta e_y &= \frac{\partial \delta v}{\partial y} - z\frac{\partial^2 w}{\partial y^2} + \frac{J_0}{a_{44}}\frac{\partial \psi}{\partial y}, \\ \delta e_{xy} &= \frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y} + J_0\left(\frac{1}{a_{55}}\frac{\partial \varphi}{\partial y} + \frac{1}{a_{44}}\frac{\partial \psi}{\partial x}\right). \end{aligned}\right\} \tag{6.11}$$

Thus, we now have values of all the strain increments, (6.9) and (6.11). Substituting the strain increments (6.11) into (6.7) we obtain the stress increments:

$$\delta\sigma_x = -z\left(b_{11}\frac{\partial^2 w}{\partial x^2} + b_{12}\frac{\partial^2 w}{\partial y^2} + 2b_{16}\frac{\partial^2 w}{\partial x\,\partial y}\right) +$$
$$+ J_0\left(\frac{b_{11}}{a_{55}}\frac{\partial\varphi}{\partial x} + \frac{b_{12}}{a_{44}}\frac{\partial\psi}{\partial y} + \frac{b_{16}}{a_{55}}\frac{\partial\varphi}{\partial y} + \frac{b_{16}}{a_{44}}\frac{\partial\psi}{\partial x}\right) +$$
$$+ b_{11}\frac{\partial\delta u}{\partial x} + b_{12}\frac{\partial\delta v}{\partial y} + b_{16}\left(\frac{\partial\delta u}{\partial y} + \frac{\partial\delta v}{\partial x}\right),$$

$$\delta\sigma_y = -z\left(b_{22}\frac{\partial^2 w}{\partial y^2} + b_{12}\frac{\partial^2 w}{\partial x^2} + 2b_{26}\frac{\partial^2 w}{\partial x\,\partial y}\right) +$$
$$+ J_0\left(\frac{b_{22}}{a_{44}}\frac{\partial\psi}{\partial y} + \frac{b_{12}}{a_{55}}\frac{\partial\varphi}{\partial x} + \frac{b_{26}}{a_{44}}\frac{\partial\psi}{\partial x} + \frac{b_{26}}{a_{55}}\frac{\partial\varphi}{\partial y}\right) +$$
$$+ b_{22}\frac{\partial\delta v'}{\partial y} + b_{12}\frac{\partial\delta u}{\partial x} + b_{26}\left(\frac{\partial\delta v}{\partial x} + \frac{\partial\delta u}{\partial y}\right),$$

$$\delta\tau_{xy} = -z\left(b_{16}\frac{\partial^2 w}{\partial x^2} + b_{26}\frac{\partial^2 w}{\partial y^2} + 2b_{66}\frac{\partial^2 w}{\partial x\,\partial y}\right) +$$
$$+ J_0\left(\frac{b_{16}}{a_{55}}\frac{\partial\varphi}{\partial x} + \frac{b_{26}}{a_{44}}\frac{\partial\psi}{\partial y} + \frac{b_{66}}{a_{55}}\frac{\partial\varphi}{\partial y} + \frac{b_{66}}{a_{44}}\frac{\partial\psi}{\partial x}\right) +$$
$$+ b_{16}\frac{\partial\delta u}{\partial x} + b_{26}\frac{\partial\delta v}{\partial y} + b_{66}\left(\frac{\partial\delta u}{\partial y} + \frac{\partial\delta v}{\partial x}\right),$$

$$\delta\tau_{xz} = f(z)\varphi, \qquad \delta\tau_{yz} = f(z)\psi.$$

In the same way as (2.2.38) – (2.2.42) it is not difficult to calculate the increments of moments and transverse stress resultants

$$\delta M_x = -\frac{h^3}{12}\left(b_{11}\frac{\partial^2 w}{\partial x^2} + b_{12}\frac{\partial^2 w}{\partial y^2} + 2b_{16}\frac{\partial^2 w}{\partial x\,\partial y}\right) +$$
$$+ J_3\left[\frac{1}{a_{55}}\left(b_{11}\frac{\partial\varphi}{\partial x} + b_{16}\frac{\partial\varphi}{\partial y}\right) + \frac{1}{a_{44}}\left(b_{12}\frac{\partial\psi}{\partial y} + b_{16}\frac{\partial\psi}{\partial x}\right)\right], \qquad (6.12)$$

$$\delta M_y = -\frac{h^3}{12}\left(b_{22}\frac{\partial^2 w}{\partial y^2} + b_{12}\frac{\partial^2 w}{\partial x^2} + 2b_{26}\frac{\partial^2 w}{\partial x\,\partial y}\right) +$$
$$+ J_3\left[\frac{1}{a_{44}}\left(b_{22}\frac{\partial\psi}{\partial y} + b_{26}\frac{\partial\psi}{\partial x}\right) + \frac{1}{a_{55}}\left(b_{12}\frac{\partial\varphi}{\partial x} + b_{26}\frac{\partial\varphi}{\partial y}\right)\right]. \qquad (6.13)$$

$$\delta H = -\frac{h^3}{12}\left(2b_{66}\frac{\partial^2 w}{\partial x\,\partial y} + b_{16}\frac{\partial^2 w}{\partial x^2} + b_{26}\frac{\partial^2 w}{\partial y^2}\right) +$$

$$+ J_3\left[\frac{1}{a_{55}}\left(b_{16}\frac{\partial \varphi}{\partial x} + b_{66}\frac{\partial \varphi}{\partial y}\right) + \frac{1}{a_{44}}\left(b_{26}\frac{\partial \psi}{\partial y} + b_{66}\frac{\partial \psi}{\partial x}\right)\right], \tag{6.14}$$

$$\delta N_1 = J_5\varphi, \qquad \delta N_2 = J_5\psi, \tag{6.15}$$

where, as usual,(2.2.43),

$$J_3 = \int_{-h/2}^{+h/2} zJ_0(z)\,dz, \qquad J_5 = \int_{-h/2}^{+h/2} f(z)\,dz. \tag{6.16}$$

The equilibrium equations of the differential element of the plate after buckling are of the following form (see (2.2.47) – (2.2.49) and (2.3))

$$\left.\begin{aligned} &\frac{\partial \delta M_x}{\partial x} + \frac{\partial \delta H}{\partial y} = \delta N_1, \qquad \frac{\partial \delta M_y}{\partial y} + \frac{\partial \delta H}{\partial x} = \delta N_2, \\ &\frac{\partial \delta N_1}{\partial x} + \frac{\partial \delta N_2}{\partial y} + T_x^0\frac{\partial^2 w}{\partial x^2} + T_y^0\frac{\partial^2 w}{\partial y^2} + 2S^0\frac{\partial^2 w}{\partial x\,\partial y} = 0, \end{aligned}\right\} \tag{6.17}$$

where T_x^0, T_y^0 and S^0 are the inplane stress resultants of the initial flat state, i.e., $T_x^0 = -ph$; $T_y^0 = -qh$; $S^0 = rh$.

Substituting the expressions for the stress and moment resultant increments from (6.12) – (6.15) into (6.17) we obtain

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \psi}{\partial y} = \frac{h}{J_5}\left(p\frac{\partial^2 w}{\partial x^2} + q\frac{\partial^2 w}{\partial y^2} - 2r\frac{\partial^2 w}{\partial x\,\partial y}\right), \tag{6.18}$$

$$b_{11}\frac{\partial^3 w}{\partial x^3} + (b_{12} + 2b_{66})\frac{\partial^3 w}{\partial x\,\partial y^2} + 3b_{16}\frac{\partial^3 w}{\partial x^2 \partial y} + b_{26}\frac{\partial^3 w}{\partial y^3} -$$

$$- \frac{12}{h^3}J_3\left\{\frac{1}{a_{55}}\left(b_{11}\frac{\partial^2 \varphi}{\partial x^2} + 2b_{16}\frac{\partial^2 \varphi}{\partial x\,\partial y} + b_{66}\frac{\partial^2 \varphi}{\partial y^2}\right) +\right.$$

$$\left. + \frac{1}{a_{44}}\left[b_{16}\frac{\partial^2 \psi}{\partial x^2} + (b_{12} + b_{66})\frac{\partial^2 \psi}{\partial x\,\partial y} + b_{26}\frac{\partial^2 \psi}{\partial y^2}\right]\right\} + \frac{12}{h^3}J_5\varphi = 0, \tag{6.19}$$

$$b_{22}\frac{\partial^3 w}{\partial y^3} + (b_{12} + 2b_{66})\frac{\partial^3 w}{\partial x^2 \partial y} + 3b_{26}\frac{\partial^3 w}{\partial x\,\partial y^2} + b_{16}\frac{\partial^3 w}{\partial x^3} -$$

$$- \frac{12}{h^3}J_3\left\{\frac{1}{a_{44}}\left(b_{22}\frac{\partial^2 \psi}{\partial y^2} + 2b_{26}\frac{\partial^2 \psi}{\partial x\,\partial y} + b_{66}\frac{\partial^2 \psi}{\partial x^2}\right) +\right.$$

$$+\frac{1}{a_{55}}\left[b_{26}\frac{\partial^2\varphi}{\partial y^2}+(b_{12}+b_{66})\frac{\partial^2\varphi}{\partial x\,\partial y}+b_{16}\frac{\partial^2\varphi}{\partial x^2}\right]\Big\}+\frac{12}{h^3}J_5\psi=0; \quad (6.20)$$

This system of equations represents the stability of a non-elastic plate.

The system of equations (6.18) – (6.20) can be reduced to one equation of sixth order by introducing the function $\Phi(x, y)$. Assuming that

$$w=\frac{144}{h^6}\Big[J_5^2-J_3J_5\Big(\frac{b_{11}}{a_{55}}+\frac{b_{66}}{a_{44}}\Big)\frac{\partial^2}{\partial x^2}-2J_3J_5\Big(\frac{b_{16}}{a_{55}}+\frac{b_{26}}{a_{44}}\Big)\frac{\partial^2}{\partial x\,\partial y}-$$
$$-J_3J_5\Big(\frac{b_{22}}{a_{44}}+\frac{b_{66}}{a_{55}}\Big)\frac{\partial^2}{\partial y^2}+\frac{J_3^2}{a_{44}a_{55}}L(b_{ik})\Big]\Phi, \quad (6.21)$$

$$\varphi=\frac{12}{h^3}\Big\{-J_5\Big[b_{11}\frac{\partial^3}{\partial x^3}+(b_{12}+2b_{66})\frac{\partial^3}{\partial x\,\partial y^2}+3b_{16}\frac{\partial^3}{\partial x^2\partial y}+$$
$$+b_{26}\frac{\partial^3}{\partial y^3}\Big]+\frac{J_3}{a_{44}}\frac{\partial}{\partial x}L(b_{ik})\Big\}\Phi, \quad (6.22)$$

$$\psi=\frac{12}{h^3}\Big\{-J_5\Big[b_{22}\frac{\partial^3}{\partial y^3}+(b_{12}+2b_{66})\frac{\partial^3}{\partial x^2\partial y}+$$
$$+3b_{26}\frac{\partial^3}{\partial x\,\partial y^2}+b_{16}\frac{\partial^3}{\partial x^3}\Big]+\frac{J_3}{a_{55}}\frac{\partial}{\partial y}L(b_{ik})\Big\}\Phi, \quad (6.23)$$

$$L(b_{ik})=c_{11}\frac{\partial^4}{\partial x^4}+2c_{16}\frac{\partial^4}{\partial x^3\partial y}+c_{12}\frac{\partial^4}{\partial x^2\partial y^2}+2c_{26}\frac{\partial^4}{\partial x\,\partial y^3}+c_{22}\frac{\partial^4}{\partial y^4}, \quad (6.24)$$

$$\left.\begin{aligned} c_{11}&=b_{11}b_{66}-b_{16}^2,\quad c_{16}=b_{11}b_{26}-b_{12}b_{16},\\ c_{22}&=b_{22}b_{66}-b_{26}^2,\quad c_{26}=b_{22}b_{16}-b_{12}b_{26},\\ c_{12}&=b_{11}b_{22}-b_{12}^2-2(b_{12}b_{66}-b_{16}b_{26}),\end{aligned}\right\} \quad (6.25)$$

equations (6.19) and (6.20) are identically satisfied. And from (6.18) a single differential equation governing the stability of a non-elastic transversely isotropic plate is derived, which accounts for transverse shear:

$$b_{11}\frac{\partial^4\Phi}{\partial x^4}+4b_{16}\frac{\partial^4\Phi}{\partial x^3\partial y}+2(b_{12}+2b_{66})\frac{\partial^4\Phi}{\partial x^2\partial y^2}+4b_{26}\frac{\partial^4\Phi}{\partial x\partial y^3}+$$
$$+b_{22}\frac{\partial^4\Phi}{\partial y^4}-\frac{J_3}{J_5}\left[\frac{c_{11}}{a_{44}}\frac{\partial^6\Phi}{\partial x^6}+2\frac{c_{16}}{c_{44}}\frac{\partial^6\Phi}{\partial x^5\partial y}+\left(\frac{c_{12}}{a_{44}}+\frac{c_{11}}{a_{55}}\right)\frac{\partial^6\Phi}{\partial x^4\partial y^2}+\right.$$
$$+2\left(\frac{c_{26}}{a_{44}}+\frac{c_{16}}{a_{55}}\right)\frac{\partial^6\Phi}{\partial x^3\partial y^3}+\left(\frac{c_{12}}{a_{55}}+\frac{c_{22}}{a_{44}}\right)\frac{\partial^6\Phi}{\partial x^2\partial y^4}+$$
$$\left.+2\frac{c_{26}}{a_{55}}\frac{\partial^6\Phi}{\partial x\partial y^5}+\frac{c_{22}}{a_{55}}\frac{\partial^6\Phi}{\partial y^6}\right]=$$
$$=\frac{h^3}{12J_5^2}\left(T_x^0\frac{\partial^2 w}{\partial x^2}+T_y^0\frac{\partial^2 w}{\partial y^2}+2S^0\frac{\partial^2 w}{\partial x\partial y}\right). \quad (6.26)$$

Examination of the system of equations (6.18) – (6.20) and (6.26) shows that externally they resemble the equations of the theory of elastic anisotropic plates.

The stability of a compressed strip made of non-elastic material. Let a long rectangular strip be simply-supported along two long sides (x = 0, x = l) and subjected to compression by a uniformly distributed (along the same sides) load of intensity p(q = 0, r = 0). The strip is of such a length that a cylindrical buckled shape is realized. (Figure 30).

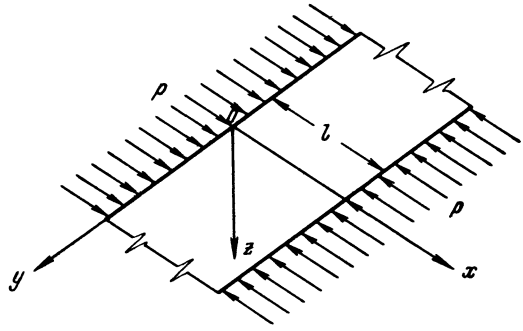

Figure 30.

From the system of equations (6.18) – (6.20) or from (6.26) it is easy to derive the stability equation for the simply-supported strip:

$$b_{11}\frac{h^3}{12}\frac{d^4w}{dx^4}=-ph\frac{d^2w}{dx^2}+ph\frac{b_{11}}{a_{55}}\frac{J_3}{J_5}\frac{d^4w}{dx^4}. \quad (6.27)$$

Assuming that

$$w = f \sin \frac{m\pi x}{l}, \tag{6.28}$$

where f is a constant; m = 1, 2, 3, ... , we satisfy the conditions of the simple supports for the strip (3.1.14), and for critical load we obtain

$$p = b_{11} \frac{h^3}{12} \frac{m^2\pi^2}{l^2 h} \left(1 + \frac{m^2\pi^2}{l^2} \frac{b_{11}}{a_{55}} \frac{J_3}{J_5}\right)^{-1}. \tag{6.29}$$

The coefficients b_{11} and a_{55}, from (6.29) and because of (6.5) – (6.8) and the initial conditions of the problem, are

$$b_{11} = E \frac{1 + \vartheta p^2}{1 - \nu^2 + \vartheta p^2 (5 - 4\nu)}, \qquad a_{55} = G'. \tag{6.30}$$

The value of θ according to (6.6), just as in an analogous problem [41], is determined from the conditions of simple stretching (compressing) of the plate material stretched within the plane of isotropy. In the case of a simple stretching, when it is assumed that $e_x = f(\sigma_x)$ and considering that the plastic deformation during stretching does not depend on the loading method and is only a function of stress intensities, i.e., $A = \Phi(T)$, this expression is obtained after some simple transformations

$$\vartheta = \frac{1}{12T^2}\left(\frac{E}{E_0} - 1\right) = \frac{1}{4p^2}\left(\frac{E}{E_0} - 1\right), \tag{6.31}$$

where E is the modulus of elasticity within the plane of isotropy; E_0 is the local or tangent modulus also within the plane of isotropy. (Remember that G' is the shear modulus within the planes Ozx and Ozy).

In order to calculate the ratio J_3/J_5 we assume that

$$f(z) = \frac{1}{2}\left(\frac{h^2}{4} - z^2\right),$$

i.e., taking into account (2.4.1) we derive on the basis of (2.2.20) and (6.16)

$$\frac{J_3}{J_5} = \frac{h^2}{10}.$$

Substituting values of θ and J_3/J_5 into (6.29) we obtain for the critical load

$$p_m^* = p_m^0 K \left[1 + \frac{m^2\pi^2}{10} \frac{h^2}{l^2} \frac{E}{G'(1 - \nu^2)} K\right]^{-1}, \tag{6.32}$$

where

$$p_m^0 = \frac{m^2\pi^2}{l^2}\frac{D}{h}, \qquad D = \frac{Eh^3}{12(1-\nu^2)},$$

$$K = \left[1 + \frac{1}{4}\left(\frac{E}{E_0} - 1\right)\right]\left[1 + \frac{5-4\nu}{4(1-\nu^2)}\left(\frac{E}{E_0} - 1\right)\right]^{-1};$$

Here p_m^0 is the value of the critical load for the elastic strip which was determined without the consideration of transverse shear.

In particular, disregarding the correction due to transverse shear, i.e., the value

$$h^* = \frac{m^2\pi^2}{10}\frac{h^2}{l^2}\frac{E}{G'(1-\nu^2)}K, \tag{6.33}$$

a critical load is obtained which is the same as that found with the classical theory, namely:

$$p_m^* = p_m^0 K. \tag{6.34}$$

Just as in Subsection 3 of Section 2, Chapter III, a few words should be said about the function f(z). It is known from many studies [42] that the distribution of shear stress τ_{xz} on yielding areas of the material degenerates into a triangular shape. Due to this fact a question is raised regarding the acceptability of the parabolic law (2.4.1) for the distibution of shear stresses in the case of a non-elastic plate. Assume that

$$f(z) = a - (2a/h)z \operatorname{sign} z.$$

Then with (2.2.20) and (6.16) we obtain $J_3/J_5 = 10h^2/96$. The value of J_3/J_5, as was expected, for this limiting case differs insignificantly ($\sim 4\%$) from the value of J_3/J_5 for the parabolic distribution. Considering that J_3/J_5 is the correction coefficient originating on account of the consideration of transverse shear, we can repeat once more that the arbitrariness in the logical selection of the function f(z) cannot introduce inadmissible errors into the stability problem of non-elastic plates when transverse shear is considered.

Let us consider one last numerical example. It is easy to note from (6.32) and (6.33) that the correction to the classical theory depends on the number of half-waves m, the relative plate thickness h/l, and the material properties of the plate material, E, E_0, G', ν, and could differ markedly from zero.

Let

$$\frac{h}{l} = 0{,}2, \qquad \nu = 0{,}5, \qquad \frac{E}{E_0} = 1{,}2, \qquad \frac{E}{G'} = 3{,}0.$$

Then

$$h^* = 0{,}138 \qquad m = 1; \qquad h^* = 0{,}552 \qquad m = 2.$$

From this we see the significance of the correction (13–55%) which enters into the value of the critical load for the stability problem of a non-elastic transversely isotropic plate when transverse shear is considered.

The influence of transverse shear could be still larger because the coefficient K, which depends on the shape of the curve $e = f(\sigma)$, can exceed unity (in this example K = 0.875), and the principal material ratio which characterizes the phenomenon related to transverse shear (E/G'), can exceed 3.

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